Arithmetical Self-Similarity

Mark Dow   Jörg Endrullis
Dimitri Hendriks   Jan Willem Klop

University of Oregon — Vrije Universiteit Amsterdam

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introduction

an infinite sequence w over $\Sigma_m = \{0, 1, \ldots, m-1\}$ is called *scale-invariant* if for all $k > 0$:

$$w/k = w + c \mod m$$

for some $c \in \Sigma_m$.

the paperfolding sequence

the paperfolding sequence $\mathsf{PF} = T(0\,?\,1?) \in \Sigma_2^\omega$ is scale-invariant !

$$
\begin{aligned}
\mathsf{PF}/1 &= 0\,0\,1\,0\,0\,1\,1\,0\,0\,0\,1\,1\,0\,1\,1\,0\,0\,0\,1\,0\,0\,1\,1\,1\,0\,0\,1\,1\,0\,1\,1\,0\,\ldots\\
\mathsf{PF}/2 &= 0\;\;0\;\;1\;\;0\;\;0\;\;1\;\;1\;\;0\;\;0\;\;0\;\;1\;\;1\;\;0\;\;1\;\;1\;\;0\;\ldots\\
\mathsf{PF}/3 &= 1\quad 1\quad 0\quad 1\quad 1\quad 0\quad 0\quad 1\quad 1\quad 1\quad \ldots\\
\mathsf{PF}/5 &= 0\qquad 0\qquad 1\qquad 0\qquad 0\qquad 1\qquad \ldots\\
&\vdots
\end{aligned}
$$

questions

- what is the underlying construction for scale-invariant sequences?
- determine the arithmetical self-similarity (= set of arith. subsequences similar) of several (classes of) streams
- relation to fractals: is scale-invariance a sufficient criterion for turtle drawings to converge to a fractal curve?

notation

- an infinite sequence (stream) w over an alphabet Σ is a function $w : \{1, 2, 3, \ldots\} \to \Sigma$. we write Σ^ω for the set of all infinite sequences over Σ.
- let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, and $m \in \overline{\mathbb{N}}$.

$$\Sigma_m = \begin{cases} \{0, 1, \ldots, m-1\} & \text{if } m \in \mathbb{N}, \\ \mathbb{N} & \text{if } m = \infty. \end{cases}$$

$$a +_m b = \begin{cases} a + b \mod m & \text{if } m \in \mathbb{N}, \\ a + b & \text{if } m = \infty. \end{cases}$$

- let $w = w_1 w_2 w_3 \ldots \in \Sigma_m$ and $c \in \Sigma m$.
  $w +_m c = (w_1 +_m c)\,(w_2 +_m c)\ \ldots$.

arithmetic subsequences

- let $w \in \Sigma^\omega$ and $a, b > 0$. the sequence $w^a_b \in \Sigma^\omega$ defined by:

$$w^a_b = w(a)\, w(a+b)\, w(a+2b)\, \dots \quad \text{, i.e.}$$
$$w^a_b(n) = w(a + (b-1)n)$$

  is called an *arithmetic subsequence* of w.
- $w/k = w^k_k = w(k)\, w(2k)\, w(3k)\, \dots$
- $(w^a_b)^c_d = w^{a+(c-1)b}_{bd}$ (and $(w/k)/\ell = w/k\ell$).

completely additive

- let $\mathbf{P} = \{2, 3, 5, \ldots\}$ denote the set of primes
- $w \in \Sigma_m^\omega$ is *additive* if for all $a, b > 0$ relatively prime

$$w(ab) = w(a) +_m w(b) \tag{1}$$

  and *completely additive* if (1) holds for all $a, b > 0$.
- e.g. logarithm is completely additive $\log 6 = \log 2 + \log 3$

p-adic valuation

- let p be prime. the *p-adic valuation* of $n > 0$ is defined by:

$$v_p(n) = \max\{a \mid p^a \text{ divides } n\}$$

- clearly v_p is completely additive: $v_p(n_1 n_2) = v_p(n_1) + v_p(n_2)$
-

$$v_p(n) = \begin{cases} v_p(n/p) + 1 & \text{if } n \equiv 0 \pmod{p} \\ 0 & \text{otherwise} \end{cases}$$

and hence

$$v_p = T_\infty(0^{p-1}\,?^{+1})$$

prime generated sequences

- completely additive sequences are determined by their values at prime positions …
- … there must be a one-one correspondence to multisets of primes !
- a *multiset* X *on a set* S is a function $X : S \to \mathbb{N}$.
  for $a \in S$, $X(a)$ is the *multiplicity of* a *in* X.
- the sequence $\Psi_m(X) \in \Sigma_m^\omega$ generated by $X : \mathbf{P} \to \mathbb{N}$ is the sum of all p-adic sequences with $p \in X$:

$$\Psi_m(X)(n) = \sum_{p \in \mathbf{P}} X(p) \cdot v_p(n) \mod m\,, \quad \text{for all } n > 0.$$

$\mathrm{PD} = \Psi_2(2) = T_2(0\,?^{+1})$

- the period doubling sequence PD is defined as 2-adic valuation modulo 2, i.e.
  $\mathrm{PD} = \Psi_2(2) = 0\,1\,0\,0\,0\,1\,0\,1\,0\,1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,1\,0\,1\,0\,0 \ldots$
- sequences generated by singleton p are Toeplitz words:
  $\Psi_m(p) = T_m(0^{p-1}\,?^{+1})$. so $\mathrm{PD} = T_2(0\,?^{+1})$
- note that $(0\,\overline{?}1)[0\,?^{+1}] = 0\,1\,0\,?$ is the more familiar pattern to define $\mathrm{PD} = T_2(0\,1\,0\,?)$

$\Psi_2(\{2,3,3\})$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
$v_2(n)$	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	...
$2 \cdot v_3(n)$	0	0	2	0	0	2	0	0	4	0	0	2	0	0	2	...
$\Psi_2(\{2,3,3\})$	0	1	2	2	0	3	0	3	4	1	0	4	0	1	2	...

$\Psi_\infty(\mathbf{P})$

$$\Psi_\infty(\mathbf{P}) = \texttt{A001222} = 0\,1\,1\,2\,1\,2\,1\,3\,2\,2\,1\,3\,1\,2\,2\,4\,1\,3\,1\,3\,2\,2\,1\,4\,2\,\ldots$$

prime generated = completely additive

- p.g. sequences are c.a.: $\Psi_m(X)(n_1 n_2) = \Psi_m(X)(n_1) +_m \Psi_m(X)(n_2)$ (just because p-adic valuation is)
- c.a. sequences are p.g.: the multiset generating a c.a. sequence w simply is $w|_{\mathbf{P}}$, i.e., the function $w : \mathbb{N}_+ \to \mathbb{N}$ restricted to the domain $\mathbf{P}$.

completely additive = scale-invariant

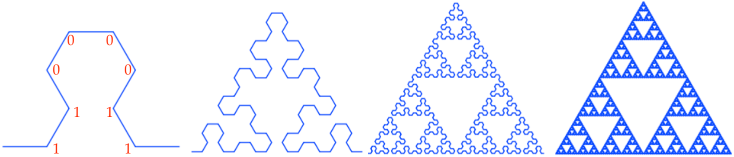
a sequence $w \in \Sigma_m^\omega$ is scale-invariant ($w/k = w +_m c$) if and only if $w - w(1)$ is completely additive

what (multi)set generates PF?

$$\mathrm{PF} = 0\,\underline{0}\,\underline{1}\,0\,\underline{0}\,1\,\underline{1}\,0\,0\,0\,\underline{1}\,1\,\underline{0}\,1\,1\,0\,\underline{0}\,0\,\underline{1}\,0\,0\,1\,\underline{1}\,1\,0\,0\,1\,1\,\underline{0}\,1\,\underline{1}\,0\,\ldots$$

$\mathrm{PF} = \Psi_2(X)$ where $X = \{p \in \mathbf{P} \mid p \equiv 3 \pmod 4\}$

The Sierpiński stream



- what is the binary sequence behind this fractal curve?
- the Toeplitz word generated by the pattern $00111100?11000011?$ is the desired sequence. This pattern can be simplified:
- the *Sierpiński stream* $\mathsf{S} \in \mathbf{2}^{\omega}$ is the Toeplitz word generated by the pattern $0\,0\,\overline{?}\,1\,1\,\overline{?}$:

$$\mathsf{S} = T_2(0\,0\,\overline{?}\,1\,1\,\overline{?})$$

- indeed $(0\,0\,\overline{?}\,1\,1\,\overline{?})[(0\,0\,\overline{?}\,1\,1\,\overline{?})] = 0\,0\,1\,1\,1\,1\,0\,0\,?\,1\,1\,0\,0\,0\,0\,1\,1\,?$
- $\mathsf{S}/2$ is generated by the set $\{p \in \mathbf{P} \mid p \equiv 2 \pmod 3)\} \cup \{3\}$

arithmetic self-similarity

- the arithmetic self-similarity of a stream $w \in \Sigma_m^\omega$ is the set of (tuples representing) arithmetic subsequences similar to w:

$$\mathcal{AS}(w) = \{\, \langle a, b, s \rangle \mid a, b \geq 1, s \in \Sigma_m, w_b^a = w +_m s \,\}$$

- arithmetic self-similarity is closed under composition of arithmetic progressions: $p \cdot q \in \mathcal{AS}(w)$, where $\langle a, b, s \rangle \cdot \langle c, d, t \rangle = \langle a + (c-1)b, bd, s +_m t \rangle$

scale-invariant = completely additive

- a sequence $w \in \Sigma_m^\omega$ is called *scale-invariant* if, for all $k \geq 1$:

$$\langle k, k, s \rangle \in \mathcal{AS}(w)\,, \quad \text{where } s = w(k) - w(1)$$

- a sequence $w \in \Sigma_m^\omega$ is scale-invariant if and only if $w - w(1)$ is completely additive.

other arith. subseqs of scale-invariant sequences?

- conjecture: let $w \in \Sigma_m$ be a non-trivial c.a. sequence. then:

$$\mathcal{AS}(w) = \{\langle k, k, w(k) \rangle \mid k \geq 1\}$$

- c.a. sequences are scale-invariant, hence $\supseteq$.
- so far $\subseteq$ only for $w = v_p$, $m = \infty$

results so far

- scale-invariant = completely additive = prime generated
- scale-invariant sequences have no self-similar arithmetic subsequences other than those with offset 0

to be investigated:

- determine the arithmetical self-similarity of several (classes of) streams.
- for instance of the Thue–Morse sequence M. conjecture: $\mathcal{AS}(\mathsf{M})$ is the closure under composition of $\{\langle 1,1,0\rangle, \langle 1,2,0\rangle, \langle 2,2,1\rangle\}$
- relation to fractals: is scale-invariance a sufficient criterion for turtle drawings to converge to a fractal curve?