Discrete Structures (5 ects)

Examination 2IT50, November 7, 2013, 9.00 - 12.00

This examination consists of 5 problems each having the same weight. The final grade is the weighted average of the result of this examination (70 %) and the average of the best two of the three interim tests (30 %).

Solutions may be given in English or Dutch.

Motivate your answers.

Problem 1.

Let R, S be two relations on a set U, of which R is transitive and S is reflexive. Prove that

$$(R;S;R)^2 \subseteq (R;S)^3.$$

Solution:

Choose $(x, y) \in (R;S;R)^2 = R;S;R;R;S;R$ be arbitrary. Then there are $z, w \in U$ such that $xR;Sz$ and $zR;Rw$ and $wS;Ry$. Since R is transitive we obtain zRw and since S is reflexive we obtain ySy. Combining $xR;Sz$, zRw, $wS;Ry$ and ySy yields $xR;S;R;S;R;Sy$, so

$$(x, y) \in R;S;R;S;R;S = (R;S)^3.$$

Hence

$$(R;S;R)^2 \subseteq (R;S)^3.$$

Problem 2.

(a) For an undirected graph in which every node has degree 3, show that the total number of nodes is always even.

Solution:

In an undirected graph (V, E) we have $\sum_{v \in V} \deg(v) = 2\#E$ is even. If $\deg(v) = 3$ for all $v \in V$, then this number is $3\#V$. If this is even then $\#V$ is even.

(b) Give an example of a Hamilton cycle in an undirected graph in which every node has degree 3.

Solution:

Take the complete graph on four nodes a, b, c, d. Then every node has degree 3, and $abcda$ is a Hamilton cycle since it contains every node exactly once.

Problem 3.

(a) Give all minimal and maximal elements of $\{5, 6, 7, 8, 9, 10, 11, 12, 15, 18\}$ with respect to the divisibility relation.

Solution:

The minimal elements are $5, 6, 7, 8, 9, 11$ since they have no other divisors than itself in the set. The maximal elements are $7, 8, 10, 11, 12, 15, 18$ since they have no other multiple than itself in the set.

(b) Let $(U, \sqsubseteq)$ be a poset, and let $A, B \subseteq U$ such that $\sup(A \cup B)$ and $\sup(A \cap B)$ exist. Prove that $\sup(A \cap B) \sqsubseteq \sup(A \cup B)$.

Solution:

First we prove that $\sup(A \cup B)$ is an upper bound of $A \cap B$. Let $x \in A \cap B$, then also $x \in A \cup B$. Since $\sup(A \cup B)$ is an upper bound of $A \cup B$ we conclude $x \sqsubseteq \sup(A \cup B)$. So $\sup(A \cup B)$ is an upper bound of $A \cap B$.

Since $\sup(A \cap B)$ is the least upper bound of $A \cap B$, we conclude

$$\sup(A \cap B) \sqsubseteq \sup(A \cup B).$$

Problem 4.

Let $(M, *, I)$ be a monoid and let $a, b \in M$ satisfy $a^2 = b^3 = I$. Prove that $(a * b * a)^6 = I$.

Solution:

We leave out parentheses which is allowed since $*$ is associative. We obtain

$$\begin{aligned} (a * b * a)^3 &= a * b * a * a * b * a * a * b * a \\ &= a * b * I * b * I * b * a \\ &= a * b * b * b * a = a * I * a = a * a = I. \end{aligned}$$

So $(a * b * a)^6 = (a * b * a)^3 * (a * b * a)^3 = I * I = I$.

Problem 5.

(a) How many injective functions exist from $\{1, 2, 3\}$ to $\{4, 5, 6, 7, 8\}$?

Solution:

For defining such an injective function f we have 5 possible choices for $f(1)$, 4 possible choices for $f(2)$, and 3 possible choices for $f(3)$. So in total $5 * 4 * 3 = 60$.

(b) Compute the greatest common divisor of $12!$ and $\binom{13}{4}$.

Solution:

$$\begin{aligned} 12! &= 1 * 2 * 3 * 4 * 5 * 6 * 7 * 8 * 9 * 10 * 11 * 12 \\ &= 2 * 3 * 2^2 * 5 * 2 * 3 * 7 * 2^3 * 3^2 * 2 * 5 * 11 * 2^2 * 3 \\ &= 2^{10} * 3^5 * 5^2 * 7 * 11 \end{aligned}$$

$\binom{13}{4} = 13 * 12 * 11 * 10/1 * 2 * 3 * 4 = 13 * 11 * 5$.

So the greatest common divisor is $5 * 11 = 55$. Note that to conclude this the precise exponents in $12!$ are not needed to be computed: the observation that 5 and 11 are divisors of $12!$ and 13 is not, is sufficient.