Stream Fusion

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Exercise:

Write a function that sums the square of all the odd integers between two arguments *m* and *n*.

Haskell solution

```
f : (Int,Int) -> Int
f = sum . map square . filter isOdd . between
  where
  square x = x * x
  isOdd x = x % 2 == 0
  between (m,n) = [m .. n]
```

Haskell solution

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  where
  square x = x * x
  isOdd x = x % 2 == 0
  between (m,n) = [m .. n]
```

Each intermediate computation creates an additional list

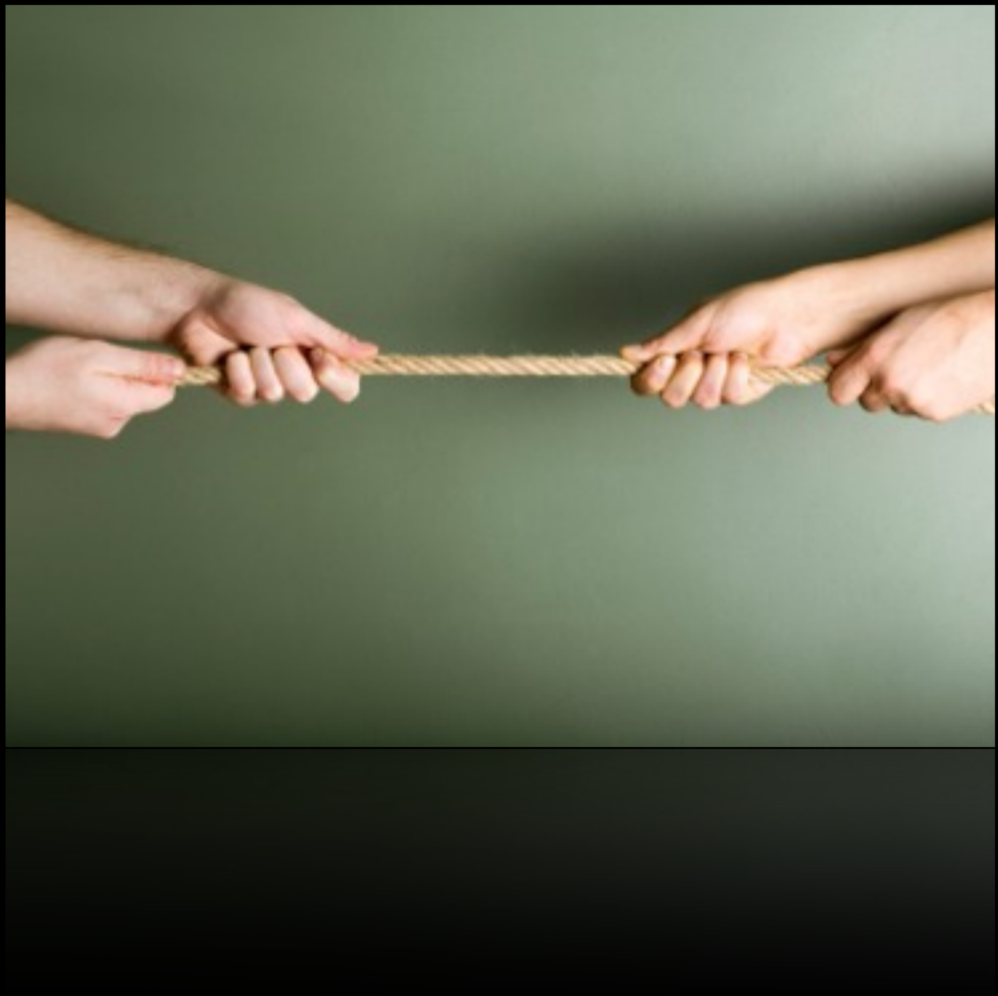
Pseudocode analogue

```
int a[]; int b[]; int c[]; int d[];
for (int i = 0; i < N; i++)
  a[i] = i
for (int j = 0; j < N; j++)
  if (a[j] % 2 == 0)
  then b[j] = a[j]
  else b[j] = 0
for (int k = 0; k < N; k++)
  c[k] = b[k] * b[k]
...
```

A “better” solution

```
f : (Int,Int) -> Int
f (m,n) = recurse m
  where
  recurse m =
    if m > n then 0
    else let rest = recurse (m + 1)
         if isOdd m then square m + rest
                    else rest
```

Composability



Efficiency

Yes – it is a problem

- Slightly changing the specification needs a complete rewrite/copy-paste of the original solution;
- Despite existing optimization techniques, the solution I will present gives significant performance improvements (up to 50% speedup on certain benchmarks).

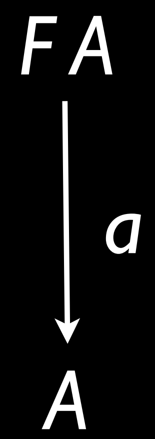
Problem:
How can we write programs that are both *efficient* and *modular*?

Solution

- Represent constituent traversals as explicit *folds* and *unfolds*;
- Exploit their universal properties to *fuse* multiple traversals into one;
- (Teach the Haskell compiler – GHC – to do this for you.)

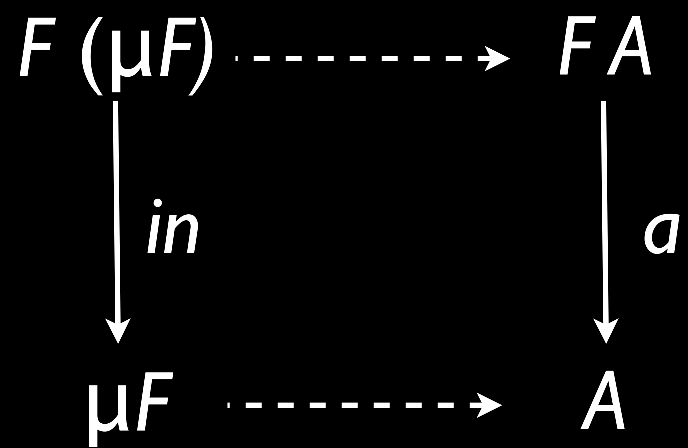
Algebras

- Given a functor $F : C \to C$, an F-algebra is a pair (a,A) where
  - A an object in C and
  - $a : FA \to A$ is a morphism in C.



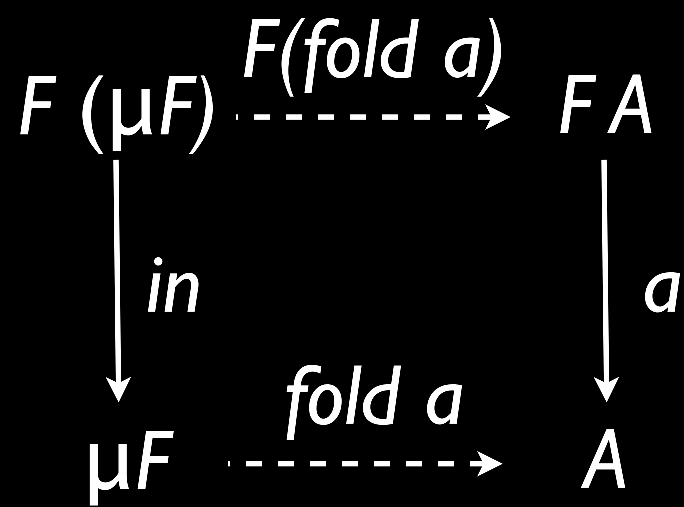
Fixed points

- You can form a category of *F*-algebras;
- If it has an initial object, call it the *least-fixed point* of *F*, usually written *(in,* μ*F*)



Folds

- As the μ*F* is initial, any algebra gives rise to a unique morphism from the least fixed-point to that algebra.



Example: sum a list

We can represent integer lists as the least-fixed point of the functor:

$L\ a = \text{Int} \times a + 1$

Or in Haskell:

```
data L a = Cons Int a | Nil
```

Example: sum a list

Taking the fixed point of the functor gives the "usual" lists:

```
data L a = Cons Int a | Nil

data Fix f = In (f (Fix f))

type List = Fix L
```

Example: sum a list

An example *L*-algebra that sums a list:

```
data L a = Cons Int a | Nil

sumAlg : L Int -> Int

sumAlg Nil = 0

sumAlg (Cons x sum) = x + sum
```

Example: sum a list

And finally, use this algebra to fold over the list, computing the sum:

```
data L a = Cons Int a | Nil

fold : (L a -> a) -> List -> a

fold f (In t) = f (mapL (fold f) t)

sum : List -> Int

sum = fold sumAlg
```

...and all this dualizes to coalgebras, greatest-fixed points, and unfolds.

Example: unfolding

Use an unfold to generate a list of all numbers between two integers *n* and *m:*

```
data L a = Cons Int a | Nil

unfold : (a -> L a) -> a -> List

betweenAlg : (Int,Int) -> L (Int,Int)

betweenAlg (m,n) = if n > m then Nil
  else Cons m (m + 1,n)
```

A technical point

- Most programming languages do not distinguish between least and greatest fixed points.
- Haskell's "ambient category" – $\mathit{CPO}_{\perp}$ – is algebraically compact and identifies the two, which some might consider a bit of a theoretical "hack".

Recursive coalgebras

A *co-algebra (C,c)* is said to be *recursive* if for every algebra *(A,a)* the equation:

$$h = a \,.\, F\, h \,.\, c$$

has a unique solution for *h*, written *hylo (a,c)*.

Such *h* (sometimes called a *hylomorphisms*) capture common divide-and-conquer algorithms.

Hylo fusion

From the universal property of hylomorphisms, we can derive for all algebras *a* and recursive coalgebras *c*:

fold a . unfold c = hylo (a,c)

Applying this rule right-to-left is called *fusion*, as two traversals are fused into one, getting rid of an intermediate data type.

Example: interval sum

Can we sum all the numbers between two arguments *n* and *m*?

First attempt:

```
fold sumAlg . unfold betweenAlg
```

But this creates an unnecessary intermediate list...

Example: interval sum

But fold/unfold law we saw previously guarantees the existence of a hylomorphism, that computes the sum directly:

```
intervalSum : (Int,Int) -> Int
intervalSum (m,n) =
  if m > n then 0
  else  m + intervalSum (m + 1,n)
```

Perspective

- So what? We already could have written that solution directly.
- Compilers tend to be very good at optimizing non-recursive functions.
- Writing functions as a composition of folds and unfolds, can generate more opportunities for optimization.

Stream fusion

- Instead of writing functions over (infinite) lists directly, write functions over their coalgebraic representation.
- (I should point out, it's not really about streams but lazy lists.)

Example: map

```
map : (a -> b) -> List a -> List b
map f Nil = Nil
map f (Cons x xs) =
  Cons (f x) (map f xs)
```

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map : (a -> b) -> List a -> List b

map f Nil = Nil

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```

How we teach our compiler that

`map f . map g = map (f . g)` ?

List Coalgebras

```
data LC a = ∃ s. (Step s a) × s

data Step s a =
    Done
  | Yield (a × s)
```

To-and-fro

It's fairly straightforward to convert between the coalgebra representation of lists and the usual inductive definition:

```
toLC : List a -> LC a

fromLC : LC a -> List a
```

Non-recursive map

```
map : (a -> b) -> List a -> List b
map f = fromLC . mapLC f . toLC

mapLC f (next,s) = (next',s)
  where
  next' Done = Done
  next' (Yield (x, s')) =
    Yield (f x,s')
```

Fusing traversals

A simple calculation teaches us:

```
map f . map g ≡
fromLC . mapLC f . toLC .fromLC . mapLC g . toLC
```

Fusing traversals

A simple calculation teaches us:

```
map f . map g ≡
fromLC . mapLC f  toLC .fromLC . mapLC g . toLC
```

Optimization stuck behind recursive functions

Cunning plan

- If only we could teach the compiler that:

  ```
  ∀c . toLC (fromLC c) = c
  ```

  we might be able to trigger more optimization.

- Using GHC's rewrite rules, we can achieve just this!

Fusing traversals

Calculating again:

```
map f . map g ≡
fromLC . mapLC f . toLC .fromLC . mapLC g . toLC ≡
fromLC . mapLC f . mapLC g . toLC ≡ (compiler magic)
fromLC . mapLC (f . g) . toLC ≡
map (f . g)
```

Result!

- At the cost of converting to-and-fro between (infinite) lists and their coalgebraic representation, we can expose more optimization opportunities to the compiler.
- But does this always work?

Filter

```
filter :
  (a -> Bool) -> List a -> List a
filter p Nil = Nil
filter p (Cons x xs) =
  if p x then Cons x (filter p xs)
  else filter p xs
```

Filter

```
filter :
  (a -> Bool) -> List a -> List a
filter p
  = fromLC . filterLC p . toLC

filterLC p (next,s) = (next',s)
  where
  next' Done = Done
  next' (Yield (x,s')) = ...
```

Filter

```
filterLC p (next,s) = (next',s)
  where
  next' (Yield (x, s')) =
    if p x then Yield (x, s')
    else filterLC s'
```

Filter

```
filterLC p (next,s) = (next',s)
  where
  next' (Yield (x, s')) =
    if p x then Yield (x, s')
    else filterLC s'
```

This function is recursive!

Stuttering

```
data LC a = ∃ s. (Step s a) × s

data Step s a =

    Done

  | Yield a × s

  | Skip s
```

Filter – revisited

```
next’ (Skip s’) = Skip s’
next’ (Yield (x, s’)) =
  if p x then Yield (x, s’)
  else Skip s’
```

Filter – revisited

```
next' (Skip s') = Skip s'
next' (Yield (x, s')) =
  if p x then Yield (x, s')
  else Skip s'
```

This function is no longer recursive!

Taking stock

- These ideas have made their way into an alternative implementation of Haskell's list library.
- Performance is 'usually better' than lists.
- Runs on co-algebraic technology.
- Many similar optimization techniques have a solid theoretical justification (foldr/build fusion, deforestation, ...)

References

- *Theory and Practice of Fusion*; Ralf Hinze, Thomas Harper, and Daniel James.
- *Stream Fusion: from Lists to Streams to Nothing at All*; Duncan Coutts, Roman Leshchinskiy, Don Stewart.