Example: Production of parts

A production system producing parts consists of 3 machining centers. The operations (and mean processing times) performed at the 3 centers are:

- Turning (70 min);
- Milling (40 min);
- Grinding (110 min).

In the first center there are 2 identical machines; in the other ones only 1 machine. Each part has to undergo the first 2 operations; only 35% the third one. Parts are transported on pallets; there are 10 pallets available. (Un)Loading is done at the Load/Unload station, which takes 25 min. It takes on average 10 minutes to transport a part to the next station.

- What is the throughput of this system?
- How does it depend on the number of pallets?

Intermezzo: Closed Queueing Networks

Consider a queueing network with

- N single-server stations, numbered $1, \ldots, N$;
- K circulating customers;
- Exponential service times, mean $1/\mu_i$ in station i;
- Random routing with routing probabilities p_{ij};

This network can be described by a Markov process with states $n = (n_1, \ldots, n_N)$ where n_i is the number of customers in station i.

Routing

Define

$$\begin{aligned} v_i &= \text{relative visit frequency to station } i \\ &= \text{expected number of visits to station } i \text{ in a cycle} \end{aligned}$$

Then the v_i's satisfy

$$v_i = \sum_{j=1}^{N} v_j p_{ji}, \qquad i = 1, \ldots, N.$$

To uniquely determine the v_i's we have to add a normzalization equation, e.g.,

$$v_1 = 1$$

(in which case a cycle is the time between two successive visits to station 1).

Product-form solution

Let $p(n)$ denote the steady-state probability of state n.

It then holds that

$$p(n) = C \cdot \left(\frac{v_1}{\mu_1}\right)^{n_1} \cdots \left(\frac{v_N}{\mu_N}\right)^{n_N}$$

where C is the normalization constant.

Using the probabilities $p(n)$ mean values like

$$\begin{aligned} L_i(K) &= \text{mean number of customers in station } i \\ S_i(K) &= \text{mean sojourn time in station } i \\ \Lambda_i(K) &= \text{throughput of station } i \\ \rho_i(K) &= \text{occupation rate of station } i \end{aligned}$$

can be computed (K indicates the dependence of these quantities on the population size).

Mean Value Analysis (MVA)

MVA is a *recursive scheme* (in the population size) for the computation of mean values. It is based on:

The Arrival Theorem:
A customer moving from station i to j sees the network in equilibrium as if he was not there (i.e., with one customer less).

Let

$$L_i^a(K) = \text{mean number of customers in station } i \text{ \emph{on arrival} of a customer}$$

Then the arrival theorem yields that

$$L_i^a(K) = L_i(K-1)$$

and hence,

$$S_i(K) = L_i^a(K)\frac{1}{\mu_i} + \frac{1}{\mu_i} = L_i(K-1)\frac{1}{\mu_i} + \frac{1}{\mu_i}$$

Together with Little's law this gives the MVA equations.

MVA relations:

$$
\begin{aligned}
S_i(K) &= L_i(K-1)\frac{1}{\mu_i} + \frac{1}{\mu_i} && \text{(Arrival relation)} \\
\Lambda_i(K) &= \frac{v_i K}{\sum_{j=1}^{N} v_j S_j(K)} && \text{(Little's law)} \\
L_i(K) &= \Lambda_i(K) S_i(K) && \text{(Little's law)}
\end{aligned}
$$

for $i = 1, \ldots, N$.

Starting with $L_i(0) = 0$ for $i = 1, \ldots, N$, these relations can be used to recursively determine $S_i(k), \Lambda_i(k), L_i(k)$ for $k = 1, \ldots, K$.

Including travel times

Suppose that it takes on average T_{ij} time units to move from station i to j.

To compute $L_i(K)$, $S_i(K)$ and $\Lambda_i(K)$ in this case, we only have to modify the relation for the througput (to take into account that some customers are 'on their way'):

$$\Lambda_i(K) = \frac{v_i K}{\sum_{j=1}^{N} v_j S_j(K) + \sum_{j=1}^{N} \sum_{l=1}^{N} p_{jl} T_{jl}}$$

The mean number of customers that is traveling from j to l is given by

$$\Lambda_j(K) p_{jl} T_{jl}$$

and the mean total number that is traveling,

$$\sum_{j=1}^{N} \sum_{l=1}^{N} \Lambda_j(K) p_{jl} T_{jl}$$