The probability of finding 5 Heads in a row

What is the probability of throwing at least k successive Heads when you throw N times with a fair coin? To find this probability we use the following approach based on Markov chains. The process of throwing a coin is said to be in state (i, n) when we still have to throw n times, and the previous i outcomes were Head, but the $i+1$th outcome was Tail. The variable i runs from 0 to k, and n runs from 0 to N. Now we define $p(i, n)$ as the probability to reach a state (k, m) for some m with $0 \leq m \leq n$, when starting in state (i, n). Then the desired probability is given by $p(0, N)$. However, we need the probabilities $p(i, n)$ for all $0 \leq i \leq k$ and $0 \leq n \leq N$ in order to recursively determine $p(0, N)$. The starting and boundary conditions for the recursion are

$$p(i, 0) = 0, \qquad 0 \leq i < k, \qquad p(k, n) = 1, \qquad n \geq 0.$$

To find a recursion for $p(i, n)$ note that, when we throw the coin and see Head, then we move to state $(i+1, n-1)$, otherwise we move to $(0, n-1)$. Hence,

$$p(i, n) = p(i+1, n-1) \cdot 1/2 + p(0, n-1) \cdot 1/2\,.$$

Now, starting with $p(i, 0) = 0$ for $i = 0, \ldots, k-1$ and $p(k, 0) = 1$, we first determine $p(i, 1)$ for $0 \leq i \leq k$, and then $p(i, 2)$ for $0 \leq i \leq k$, and so on, till we eventually find $p(0, N)$. This recursive approach, which is useful in many applications, can be easily implemented (see e.g. C-code). For 20 trials we obtain that the probability of throwing at least five successive Heads is equal to 0.2499.