9 The $M/G/1$ system with setup times

In this chapter we treat some variations of the $M/G/1$ system and we demonstrate that the mean value technique is a powerful technique to evaluate mean performance characteristics in these systems.

Consider a single machine where jobs are being processed in order of arrival and suppose that it is expensive to keep the machine in operation while there are no jobs. Therefore the machine is turned off as soon as the system is empty. When a new job arrives the machine is turned on again, but it takes some setup time till the machine is ready for processing. So turning off the machine leads to longer production leadtimes. In the following sections we investigate for some simple models how much longer.

9.1 Exponential processing and setup times

Suppose that the jobs arrive according to a Poisson stream with rate λ and that the processing times are exponentially distributed with mean $1/\mu$. For stability we have to require that $\rho = \lambda/\mu < 1$. The setup time of the machine is also exponentially distributed with mean $1/\theta$. We now wish to determine the mean production lead time $E(S)$ and the mean number of jobs in the system $E(L)$. These means can be determined by using the mean value technique.

To derive an equation for the mean production lead time, i.e., *the arrival relation*, we evaluate what is seen by an arriving job. We know that the mean number of jobs in the system found by an arriving job is equal to $E(L)$ and each of them (also the one being processed) has an exponential (residual) processing time with mean $1/\mu$. With probability $1 - \rho$ the machine is not in operation on arrival, in which case the job also has to wait for the (residual) setup phase with mean $1/\theta$. Further the job has to wait for its own processing time. Hence

$$E(S) = (1-\rho)\frac{1}{\theta} + E(L)\frac{1}{\mu} + \frac{1}{\mu}$$

and together with Little's law,

$$E(L) = \lambda E(S)$$

we immediately find

$$E(S) = \frac{1/\mu}{1-\rho} + \frac{1}{\theta}\,.$$

So the mean production lead time is equal to the one in the system where the machine is always on, plus an extra delay caused by turning off the machine when there is no work. Surprisingly, the mean extra delay is not less but exactly equal to the mean setup time. In fact, it can be shown that the extra delay has the same distribution as the setup time, i.e., it is exponentially distributed with mean $1/\theta$.

9.2 General processing and setup times

We now consider the model with generally distributed processing times and generally distributed setup times. The arrivals are still Poisson with rate λ. The first and second moment of the processing time are denoted by $E(B)$ and $E(B^2)$ respectively, $E(T)$ and $E(T^2)$ are the first and second moment of the setup time. For stability we require that $\rho = \lambda E(B) < 1$. Below we demonstrate that also in this more general setting the mean value technique can be used to find the mean production lead time. We first determine the mean waiting time. Then the mean production lead time is found afterwards by adding the mean processing time.

The mean waiting time $E(W)$ of a job satisfies

$$\begin{aligned} E(W) &= E(L^q)E(B) + \rho E(R_B) \\ &+ P(\text{Machine is off on arrival})E(T) \\ &+ P(\text{Machine is in setup phase on arrival})E(R_T), \end{aligned} \tag{1}$$

where $E(R_B)$ and $E(R_T)$ denote the mean residual processing and residual setup time, so

$$E(R_B) = \frac{E(B^2)}{2E(B)}, \qquad E(R_T) = \frac{E(T^2)}{2E(T)}.$$

To find the probability that on arrival the machine is off (i.e. not working *and* not in the setup phase), note that by PASTA, this probability is equal to the fraction of time that the machine is off. Since a period in which the machine is not processing jobs consists of an interarrival time followed by a setup time, we have

$$P(\text{Machine is off on arrival}) = (1-\rho)\frac{1/\lambda}{1/\lambda + E(T)}.$$

Similarly we find

$$P(\text{Machine is in setup phase on arrival}) = (1-\rho)\frac{E(T)}{1/\lambda + E(T)}.$$

Substituting these relations into (1) and using Little's law stating that

$$E(L^q) = \lambda E(W)$$

we finally obtain that

$$E(W) = \frac{\rho E(R_B)}{1-\rho} + \frac{1/\lambda}{1/\lambda + E(T)}E(T) + \frac{E(T)}{1/\lambda + E(T)}E(R_T).$$

Note that the first term at the right-hand side is equal to the mean waiting time in the $M/G/1$ without setup times, the other terms (which are more complicated than in the exponential case) express the extra delay due to the setup times. Finally, the mean production lead time $E(S)$ follows by simply adding the mean processing time $E(B)$ to $E(W)$.

9.3 Threshold setup policy

To save setup costs, a natural extension to the setup policy in the previous section is the one in which the machine is switched on when the number of jobs in the system reaches some threshold value, N say. This situation can be analyzed along the same lines as in the previous section. The mean waiting time now satisfies

$$\begin{aligned} E(W) &= E(L^q)E(B) + \rho E(R_B) \\ &+ \sum_{i=1}^{N} P(\text{Arriving job is number } i)\left(\frac{N-i}{\lambda} + E(T)\right) \\ &+ P(\text{Machine is in setup phase on arrival})E(R_T), \end{aligned} \tag{2}$$

The probability that an arriving job is the i-th one in a cycle can be determined as follows. A typical production cycle is displayed in figure 1.

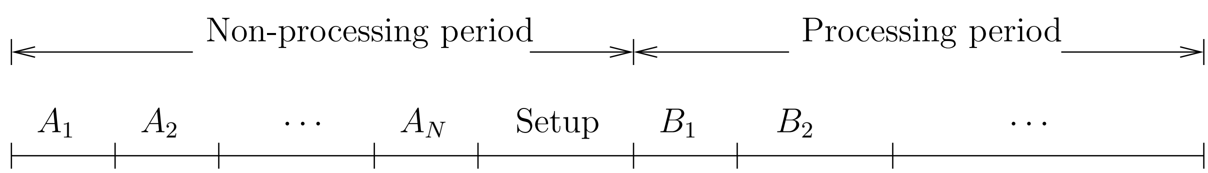

Figure 1: Production cycle in case the machine is switched on when there are N jobs

The probability that a job arrives in a non-processing period is equal to $1 - \rho$. Such a period now consists of N interarrival times followed by a setup time. Hence, the probability that a job is the i-th one, given that the job arrives in a non-processing period, is equal to $1/\lambda$ divided by the mean length of a non-processing period. So

$$P(\text{Arriving job is number } i) = (1-\rho)\frac{1/\lambda}{N/\lambda + E(T)}, \qquad i = 1, \ldots, N,$$

and similarly,

$$P(\text{Machine is in setup phase on arrival}) = (1-\rho)\frac{E(T)}{N/\lambda + E(T)}.$$

Substituting these relations into (2) we obtain, together with Little's law, that

$$E(W) = \frac{\rho E(R_B)}{1-\rho} + \frac{N/\lambda}{N/\lambda + E(T)}\left(\frac{N-1}{2\lambda} + E(T)\right) + \frac{E(T)}{N/\lambda + E(T)}E(R_T).$$