INSTRUCTION WEEK 1, 5 FEBRUARY 2016

Exercise 1. The binomial distribution with parameters n and p is the discrete probability distribution of the number of successes in a sequence of n independent experiments, where each experiment yields success with probability p. Let X be such a binomially distributed random variable with parameters n and p. The probability mass function of X is given by

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \leq k \leq n,$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. In this case, one should read $\mathbb{P}(X = k)$ as the probability that out of n experiments, k were a success. Determine $\mathbb{E}[X] = \sum_{k=0}^{n} k\mathbb{P}(X = k)$.
Hint. At some point you will need to use the binomial theorem: $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Exercise 2. Let X_i be an exponentially distributed random variable with parameter λ_i, $i = 1, 2$.

(a) Determine $\mathbb{P}(X_1 > s + t \mid X_1 > s)$. Can you explain this in words?

(b) Determine $\mathbb{P}(X_1 \leq X_2)$.

(c) Determine $\mathbb{P}(\min(X_1, X_2) \leq t)$. What type of random variable is $\min(X_1, X_2)$? What are its parameter(s)?

Exercise 3. Consider two parallel processors, 1 and 2. Job A_i is being processed by processor $i = 1, 2$. The processing times of jobs at processor i are exponentially distributed with parameters λ_i, $i = 1, 2$. Job A_3 is waiting in line and will be processed by the processor that completes its current job first.

(a) Let $\lambda_1 = \lambda_2 = \lambda$. Without calculations, determine the probability that job A_3 is the last of the three jobs to be completed.

(b) Determine the probability that job A_3 is the last of the three jobs to be completed for arbitrary values of λ_1 and λ_2.

Hint. Use (a) and (b) from exercise 2 above.

Exercise 4. Let $X_1, X_2, \ldots$ be a sequence of independent, exponentially distributed random variables with common parameter λ.

(a) Determine the probability density function $f_{X_1+X_2}(\cdot)$ of $X_1 + X_2$.

(b) Use induction on n to show that the probability density function $f_{S_n}(\cdot)$ of $S_n := \sum_{i=1}^{n} X_i$ is given by

$$f_{S_n}(x) = \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!}. \tag{1}$$

Hint. Use the following property: If X and Y are independent, non-negative random variables with probability density functions $f(\cdot)$ and $g(\cdot)$, then the probability density function of $X + Y$ equals $f_{X+Y}(x) = \int_0^x f(u)g(x-u)\mathrm{d}u$.

Exercise 5. Consider a Poisson process $N(t)$ with parameter λ. Show that the expected value $\mathbb{E}[N(t)] = \lambda t$.
Hint. $\sum_{n=0}^{\infty} x^n/n! = \mathrm{e}^x$.

Exercise 6. Let $X_1, X_2, \dots$ be a sequence of independent, exponentially distributed random variables with parameter λ, the same as in exercise 4. Then, $S_n := \sum_{i=1}^{n} X_i$ has an Erlang-n distribution with mean n/λ, see Section 2.2.4. Let $f_{S_n}(x)$ be the probability density function of S_n and let $F_{S_n}(x) = \mathbb{P}(S_n \leq x)$ be the cumulative density function of S_n.
Show that $f_{S_n}(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{S_n}(x)$.

Exercise 7. Patients arrive to a hospital according to a Poisson process $\{N(t),\ t \geq 0\}$ with parameter λ patients per hour. The doctor starts to examine patients only when the third patient arrives. Compute the probability that after 1.5 hours, the doctor has not started examining patients.