Stochastic performance modelling (2WB60)

Instruction week 4, 4 March 2016

Exercise 1. A total of N lorries drive back and forth between a loading platform and an unloading platform of a transshipment terminal. Lorries queue up at the loading platform to be loaded by a (single) crane, with exponentially distributed loading times (per lorry) with parameter λ, all loading times being independent of each other. Similarly, items are removed from the lorries by a single crane at the unloading platform, with (independent) exponentially distributed unloading times with parameter μ, $\mu \neq \lambda$. Lorry driving times between the two platforms may be neglected (with respect to loading and unloading times), and it is further assumed that there is an abundance of items at the loading platform and that the system is in equilibrium.

(a) What is the probability that k of the N lorries are at the loading platform? (**A**: $p_k = \beta^k \frac{1-\beta}{1-\beta^{N+1}}$ with $\beta = \mu/\lambda$)
Hint. Draw a transition rate diagram for the number of lorries at the loading platform. Use $\sum_{k=0}^{N} x^k = \frac{1-x^{N+1}}{1-x}$ if $|x| \neq 1$.

(b) The dynamics of this model are identical to that of a well-known queueing model. Describe that queueing model.

Answer.

(a) The transition rate diagram of the number of lorries at the loading platform is shown below.

The balance equations are given by $p_k\lambda = p_{k-1}\mu$, $k = 1, 2, \ldots, N$ and thus $p_k = (\mu/\lambda)^k p_0$, $k = 0, 1, \ldots, N$. Together with the normalization condition $\sum_{k=0}^{N} p_k = 1$ we obtain $p_0 = \frac{1-\mu/\lambda}{1-(\mu/\lambda)^{N+1}}$.

(b) It is an $M/M/1/N$ queue with arrival rate μ and departure rate λ.

Exercise 2. A repair facility for automatic copiers has three repairmen. Repair requests occur according to a Poisson process with a rate of $\lambda = 5$ per day. The repair times are exponentially distributed with a mean of $1/\mu = 1/3$ day. The system is assumed to be in equilibrium.

(a) Determine the transition rate diagram.

(b) What is the mean number of waiting copiers at the repair facility (mean queue length). (**A**: 0.3747)

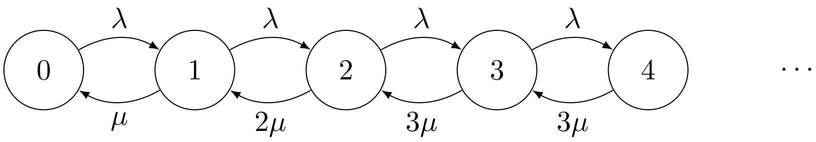
(c) What is the mean waiting time of a copier at the repair facility? (**A**: 0.0749)

(d) What is the mean number of active repairmen? (**A**: 5/3)

(e) What is the utilization factor of a repairman, i.e., the fraction of time that the repairman is busy? (**A**: 5/9)

Answer.

(a) This can be modeled as an $M/M/3$ model with arrival rate $\lambda = 5$, service rate $\mu = 3$ and load $\rho = \lambda/(3\mu) = 5/9$.



(b) The probability that a copier has to wait is denoted by $\Pi_W \approx 0.2998$ and can be calculated from equation (4.5) of the lecture notes. The mean queue length is $\mathbb{E}[L^q] = \Pi_W \frac{\rho}{1-\rho} \approx 0.3747$, see equation (4.6).

(c) Using Little's law we get $\mathbb{E}[W] = \mathbb{E}[L^q]/\lambda = \Pi_W \frac{1}{1-\rho}\frac{1}{3\mu} \approx 0.0749$.

(d) One can immediately deduce this from the fact that $c\rho$ is the mean number of active servers in an $M/M/c$. So, we have $3\rho = 5/3$.

Alternative: In state 1, 1 repairman is active, in state 2, 2 repairmen are active and in states $i \geq 3$, all three repairmen are active. So, we get that the number of active repairmen is equal to $p_1 + 2p_2 + 3\sum_{i=3}^{\infty} p_i = p_1 + 2p_2 + 3(1 - p_0 - p_1 - p_2) = 3 - 3p_0 - 2p_1 - p_2 = 5/3$, where p_k are obtained from Section 4.4.1.

(e) The fraction of time a repairman is busy is equal to $\rho = 5/9$.

Alternative: In state 1, 1 out of 3 repairmen are busy, in state 2, 2 out of 3 repairmen are busy and in states $i \geq 3$, all three repairmen are active. So, we get $p_1/3 + 2p_2/3 + \sum_{i=3}^{\infty} p_i = p_1/3 + 2p_2/3 + (1 - p_0 - p_1 - p_2) = 1 - p_0 - 2p_1/3 - p_2/3 = 5/9$.

Exercise 3. Consider an $M/G/1$ queue with arrival rate λ, mean service time $\mathbb{E}[B]$ and second moment of the service times $\mathbb{E}[B^2]$.

(a) Suppose $\lambda = 3/2$, $\mathbb{E}[B] = 1/2$ and $\mathbb{E}[B^2] = 1/2$. Determine the expected values of the waiting time, the sojourn time, the queue length, the number of customers in the system and the length of the busy period. (**A**: $\mathbb{E}[W] = 3/2$, $\mathbb{E}[S] = 2$, $\mathbb{E}[L^q] = 9/4$, $\mathbb{E}[L] = 3$, $\mathbb{E}[P] = 2$)

(b) Suppose that from measurements it is observed that the expected waiting time equals 5, that the traffic load ρ equals 2/3, and that the mean number of customers in the system is 8. Determine the arrival rate λ and the first two moments of the service time distribution. What are the expected values of the queue length and the sojourn time? (**A**: $\lambda = 22/15$, $\mathbb{E}[B] = 5/11$, $\mathbb{E}[B^2] = 25/11$, $\mathbb{E}[L^q] = 22/3$, $\mathbb{E}[S] = 60/11$)

Answer.

(a) $\rho = \lambda\mathbb{E}[B] = 3/4$, $\mathbb{E}[W] = \lambda\mathbb{E}[B^2]/(2(1-\rho)) = 3/2$, $\mathbb{E}[S] = \mathbb{E}[W] + \mathbb{E}[B] = 2$, $\mathbb{E}[L^q] = \lambda^2\mathbb{E}[B^2]/(2(1-\rho)) = 9/4$, $\mathbb{E}[L] = \mathbb{E}[L^q] + \rho = 3$, and $\mathbb{E}[P] = \mathbb{E}[B]/(1-\rho) = 2$.

(b) $\mathbb{E}[L] = \mathbb{E}[L^q] + \rho \Rightarrow \mathbb{E}[L^q] = \mathbb{E}[L] - \rho = 22/3$. $\mathbb{E}[L^q] = \lambda\mathbb{E}[W] \Rightarrow \lambda = \mathbb{E}[L^q]/\mathbb{E}[W] = 22/15$. $\rho = \lambda\mathbb{E}[B] \Rightarrow \mathbb{E}[B] = \rho/\lambda = 5/11$. $\mathbb{E}[W] = \lambda\mathbb{E}[B^2]/(2(1-\rho)) \Rightarrow \mathbb{E}[B^2] = \mathbb{E}[W](2(1-\rho)/\lambda = 25/11$. $\mathbb{E}[S] = \mathbb{E}[W] + \mathbb{E}[B] = 60/11$.