Instruction week 2, 19 February 2016

**Exercise 1.** Consider a Markov chain with state space $\{1, 2, 3\}$ and transition probabilities $p_{12} = 1$, $p_{21} = 1/2$, $p_{23} = 1/2$, $p_{32} = 1/2$ and $p_{33} = 1/2$. Determine the steady-state distribution of this Markov chain. (**A**: $\pi = (1, 2, 2)/5$)

**Answer.** Use $\pi = \pi P$ with $\pi = (\pi_1, \pi_2, \pi_3)$ a row vector, and the normalization condition $\pi_1 + \pi_2 + \pi_3 = 1$ to obtain $\pi = (1/5, 2/5, 2/5)$.

**Exercise 2.** A processor is inspected weekly in order to determine its condition. The condition of the processor can either be *perfect*, *good*, *reasonable*, or *bad*. A new processor is still perfect after one week with probability 0.7, with probability 0.2 the state is good, and with probability 0.1 it is reasonable. A processor in good condition is still good after one week with probability 0.6, reasonable with probability 0.2, and bad with probability 0.2. A processor in reasonable condition is still reasonable after one week with probability 0.5 and bad with probability 0.5. A bad processor must be repaired. The repair takes one week, after which the processor is again in perfect condition.

(a) Formulate a Markov chain that describes the state of the machine, and draw the corresponding transition diagram.

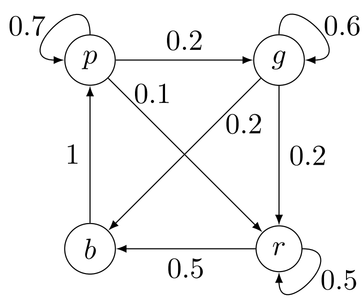
(b) Determine the steady-state distribution of the Markov chain. (**A**: $\pi = (10, 5, 4, 3)/22$)

**Answer.**

(a) The state space is given by $S = \{p, g, r, b\}$ and the transition probability matrix is

$$P = \begin{pmatrix} 0.7 & 0.2 & 0.1 & 0 \\ 0 & 0.6 & 0.2 & 0.2 \\ 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The transition diagram is as follows




(b) Use $\pi = \pi P$ with $\pi = (\pi_p, \pi_g, \pi_r, \pi_b)$ a row vector, and the normalization condition $\pi_p + \pi_g + \pi_r + \pi_b = 1$. This gives $\pi = (10/22, 5/22, 4/22, 3/22)$.

**Exercise 3.** Consider the Markov chain described in Example 3.3.

(a) Derive the balance equations

$$\pi_i = \pi_{i+1}p + \pi_{i-1}q, \quad i \geq 1, \tag{1}$$

and

$$\pi_0 = \pi_0 p + \pi_1 p, \tag{2}$$

by using the fact that there is a balance between the probability of moving to state $i$ and the probability of moving away from state $i$ ('in = out').

(b) Show that the limiting distribution satisfies $\pi_i = (q/p)^i(1 - q/p), \; i \geq 0$.

(c) Also derive the limiting distribution using probability generating functions.

*Hint.* Let $L$ be the number of cells in the buffer. The probability generating function of $L$ is given by

$$P_L(z) = \mathbb{E}[z^L] = \sum_{k=0}^{\infty} \mathbb{P}(L = k)z^k = \sum_{k=0}^{\infty} \pi_k z^k. \tag{3}$$

Think about multiplying the equations by a certain factor and then summing over all equations. The probability generating function of a geometric random variable $X$ with parameter $p$ is given by

$$P_X(z) = \mathbb{E}[z^X] = \sum_{k=0}^{\infty} \mathbb{P}(X = k)z^k = \frac{1-p}{1-pz}. \tag{4}$$

**Answer.**

(a) The probability of moving away from state $i, i \geq 1$ is 1. The probability of moving to state $i$ is $\pi_{i+1}p + \pi_{i-1}q$. For state 0 we have that the probability of moving away is 1 and the probability of arriving to state 0 is $\pi_0 p + \pi_1 p$.

(b) From (2) we get $\pi_1 = (q/p)\pi_0$. Then, from (1) for $i = 1$ we get $\pi_2 = (q/p)\pi_1$ and by continuing like this we have $\pi_i = (q/p)\pi_{i-1} = \cdots = (q/p)^i \pi_0$. With the normalization condition $\sum_{i=0}^{\infty} \pi_i = 1$ we get the limiting distribution as specified.

(c) Let $L$ be the number of cells in the buffer. Then we are interested in $P_L(z) = \sum_{i=0}^{\infty} \pi_i z^i$. Multiply (1) by $z^i$ and sum over all $i \geq 1$ to obtain

$$\begin{aligned}
\sum_{i=1}^{\infty} \pi_i z^i &= p \sum_{i=1}^{\infty} \pi_{i+1} z^i + q \sum_{i=1}^{\infty} \pi_{i-1} z^i, \\
\Rightarrow P_L(z) - \pi_0 &= p/z \sum_{j=2}^{\infty} \pi_j z^j + qz \sum_{j=0}^{\infty} \pi_j z^j, \\
&= p/z\big(P_L(z) - \pi_0 - \pi_1 z\big) + qzP_L(z), \\
\Rightarrow (1 - qz - p/z)P_L(z) &= \pi_0(1 - p/z) - \pi_1 p.
\end{aligned}$$

Now use that $\pi_1 = q/p\pi_0$ and $q = 1 - p$ to obtain

$$P_L(z) = \frac{\pi_0 p(1 - 1/z)}{1 - qz - p/z} = \frac{\pi_0 p(1 - 1/z)}{p(1 - 1/z)(1 - (q/p)z)} = \frac{\pi_0}{1 - (q/p)z}.$$

We know that $P_L(1) = \sum_{i=0}^{\infty} \pi_i = 1$, so we get $\pi_0 = 1 - q/p$ and thus

$$P_L(z) = \frac{1 - q/p}{1 - (q/p)z} = \sum_{i=0}^{\infty} (1 - q/p)(q/p)^i z^i.$$

Thus, $L$ is geometrically distributed with parameter $q/p$.