INSTRUCTION WEEK 1, 5 FEBRUARY 2016

Exercise 1. The binomial distribution with parameters n and p is the discrete probability distribution of the number of successes in a sequence of n independent experiments, where each experiment yields success with probability p. Let X be such a binomially distributed random variable with parameters n and p. The probability mass function of X is given by

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \leq k \leq n,$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. In this case, one should read $\mathbb{P}(X = k)$ as the probability that out of n experiments, k were a success. Determine $\mathbb{E}[X] = \sum_{k=0}^{n} k\mathbb{P}(X = k)$.
Hint. At some point you will need to use the binomial theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Answer.

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
&= n \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} p^k (1-p)^{n-k} \\
&= n \sum_{k=1}^{n} \binom{n-1}{k-1} p^k (1-p)^{n-k} \\
&= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)} \\
&= np \sum_{l=0}^{n-1} \binom{n-1}{l} p^l (1-p)^{(n-1)-l} \\
&= np(p + (1-p))^{n-1} = np1^{n-1} = np,
\end{aligned}$$

where in line 1 we used that for $k = 0$ the summand is 0; in line 3 we used $n! = n(n-1)!$, $\frac{k}{k!} = \frac{1}{(k-1)!}$ and $n - k = (n-1) - (k-1)$; in line 6 we set $l = k - 1$; and in line 7 we used the binomial theorem.

Exercise 2. Let X_i be an exponentially distributed random variable with parameter λ_i, $i = 1, 2$.

(a) Determine $\mathbb{P}(X_1 > s + t \mid X_1 > s)$. Can you explain this in words?

(b) Determine $\mathbb{P}(X_1 \leq X_2)$.

(c) Determine $\mathbb{P}(\min(X_1, X_2) \leq t)$. What type of random variable is $\min(X_1, X_2)$? What are its parameter(s)?

Answer.

(a) We use $\mathbb{P}(A \mid B) = \mathbb{P}(A \cap B)/\mathbb{P}(B)$ to obtain

$$\mathbb{P}(X_1 > s+t \mid X_1 > s) = \frac{\mathbb{P}(X_1 > s+t \cap X_1 > s)}{\mathbb{P}(X_1 > s)} = \frac{\mathbb{P}(X_1 > s+t)}{\mathbb{P}(X_1 > s)} = \frac{\mathrm{e}^{-\lambda_1(s+t)}}{\mathrm{e}^{-\lambda_1 s}} = \mathrm{e}^{-\lambda_1 t}.$$

(b) We use $\mathbb{P}(A \leq B) = \int_0^\infty f_A(x)\mathbb{P}(B \geq x)\mathrm{d}x$ to obtain

$$\begin{aligned}\mathbb{P}(X_1 \leq X_2) &= \int_0^\infty f_{X_1}(x)\mathbb{P}(X_2 \geq x)\mathrm{d}x = \int_0^\infty \lambda_1 \mathrm{e}^{-\lambda_1 x}\mathrm{e}^{-\lambda_2 x}\mathrm{d}x \\ &= \lambda_1 \int_0^\infty \mathrm{e}^{-(\lambda_1+\lambda_2)x}\mathrm{d}x = \frac{\lambda_1}{\lambda_1+\lambda_2}.\end{aligned}$$

(c) We find that the minimum of two exponential random variables is again an exponential random variable.

$$\begin{aligned}\mathbb{P}(\min(X_1, X_2) \leq t) &= 1 - \mathbb{P}(\min(X_1, X_2) > t) = 1 - \mathbb{P}(X_1 > t)\mathbb{P}(X_2 > t) \\ &= 1 - \mathrm{e}^{-\lambda_1 t}\mathrm{e}^{-\lambda_2 t} = 1 - \mathrm{e}^{-(\lambda_1+\lambda_2)t}.\end{aligned}$$

Exercise 3. Consider two parallel processors, 1 and 2. Job A_i is being processed by processor $i = 1, 2$. The processing times of jobs at processor i are exponentially distributed with parameters $\lambda_i,\ i = 1, 2$. Job A_3 is waiting in line and will be processed by the processor that completes its current job first.

(a) Let $\lambda_1 = \lambda_2 = \lambda$. Without calculations, determine the probability that job A_3 is the last of the three jobs to be completed.

(b) Determine the probability that job A_3 is the last of the three jobs to be completed for arbitrary values of λ_1 and λ_2.

Hint. Use (a) and (b) from exercise 2 above.

Answer.

(a) Due to the memoryless property of the exponential distribution, see exercise 2(a), we have that this probability is 1/2.

(b) There are two scenarios in which job A_3 finishes last, namely job A_1 finishes first and then job A_2 finishes or vice versa. So, by using exercise 2(b), we have

$$\begin{aligned}\mathbb{P}(\text{Job } A_3 \text{ finishes last}) &= \mathbb{P}(\text{Job } A_1 \text{ finishes and then } A_2) + \mathbb{P}(\text{Job } A_2 \text{ finishes and then } A_1) \\ &= \mathbb{P}(A_1 \leq A_2)\mathbb{P}(A_2 \leq A_3) + \mathbb{P}(A_2 \leq A_1)\mathbb{P}(A_1 \leq A_3) \\ &= \frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\lambda_2}{\lambda_1+\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2}\frac{\lambda_1}{\lambda_1+\lambda_2} = 2\frac{\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2}.\end{aligned}$$

Exercise 4. Let $X_1, X_2, \ldots$ be a sequence of independent, exponentially distributed random variables with common parameter λ.

(a) Determine the probability density function $f_{X_1+X_2}(\cdot)$ of $X_1 + X_2$.

(b) Use induction on n to show that the probability density function $f_{S_n}(\cdot)$ of $S_n := \sum_{i=1}^{n} X_i$ is given by

$$f_{S_n}(x) = \lambda \mathrm{e}^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!}. \tag{1}$$

Hint. Use the following property: If X and Y are independent, non-negative random variables with probability density functions $f(\cdot)$ and $g(\cdot)$, then the probability density function of $X+Y$ equals $f_{X+Y}(x) = \int_0^x f(u)g(x-u)\mathrm{d}u$.

Answer.

(a) Use the hint so that

$$\begin{aligned} f_{X_1+X_2}(x) &= \int_0^x f_{X_1}(u) f_{X_2}(x-u)\mathrm{d}u = \int_0^x \lambda \mathrm{e}^{-\lambda u} \lambda \mathrm{e}^{-\lambda(x-u)} \mathrm{d}u \\ &= \lambda^2 \mathrm{e}^{-\lambda x} \int_0^x \mathrm{d}u = \lambda^2 x \mathrm{e}^{-\lambda x}. \end{aligned}$$

(b) For $n = 1$ we have $f_{S_1}(x) = \lambda \mathrm{e}^{-\lambda x}$. Let us now show that if (1) holds for n, then (1) also holds for $n+1$. Assume that $f_{S_n}(\cdot)$ satisfies (1). Then,

$$\begin{aligned} f_{S_{n+1}}(x) = f_{S_n + X_{n+1}}(x) &= \int_0^x f_{S_n}(u) f_{X_{n+1}}(x-u)\mathrm{d}u \\ &= \int_0^x \lambda \mathrm{e}^{-\lambda u} \frac{(\lambda u)^{n-1}}{(n-1)!} \lambda \mathrm{e}^{-\lambda(x-u)} \mathrm{d}u \\ &= \lambda \mathrm{e}^{-\lambda x} \frac{\lambda^n}{(n-1)!} \int_0^x u^{n-1} \mathrm{d}u \\ &= \lambda \mathrm{e}^{-\lambda x} \frac{(\lambda x)^n}{n!}. \end{aligned}$$

Exercise 5. Consider a Poisson process $N(t)$ with parameter λ. Show that the expected value $\mathbb{E}[N(t)] = \lambda t$.
Hint. $\sum_{n=0}^{\infty} x^n/n! = \mathrm{e}^x$.

Answer.

$$\begin{aligned} \mathbb{E}[N(t)] &= \sum_{k=0}^{\infty} k \mathbb{P}(N(t) = k) = \sum_{k=1}^{\infty} k \frac{(\lambda t)^k}{k!} \mathrm{e}^{-\lambda t} \\ &= \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!} \mathrm{e}^{-\lambda t} = \lambda t \mathrm{e}^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^{k-1}}{(k-1)!} \\ &= \lambda t \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} = \lambda t \mathrm{e}^{-\lambda t} \mathrm{e}^{\lambda t} = \lambda t. \end{aligned}$$

Exercise 6. Let $X_1, X_2, \ldots$ be a sequence of independent, exponentially distributed random variables with parameter λ, the same as in exercise 4. Then, $S_n := \sum_{i=1}^{n} X_i$ has an Erlang-n distribution with mean n/λ, see Section 2.2.4. Let $f_{S_n}(x)$ be the probability density function of S_n and let $F_{S_n}(x) = \mathbb{P}(S_n \leq x)$ be the cumulative density function of S_n.
Show that $f_{S_n}(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{S_n}(x)$.

Answer.

$$
\begin{aligned}
f_{S_n}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}F_{S_n}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(S_n \le x) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\Big(1 - \sum_{i=0}^{n-1}\frac{(\lambda x)^i}{i!}\mathrm{e}^{-\lambda x}\Big) \\
&= -\frac{\mathrm{d}}{\mathrm{d}x}\sum_{i=0}^{n-1}\frac{(\lambda x)^i}{i!}\mathrm{e}^{-\lambda x} \\
&= -\sum_{i=0}^{n-1}\frac{\lambda^i}{i!}\frac{\mathrm{d}}{\mathrm{d}x}x^i\mathrm{e}^{-\lambda x} \\
&= -\sum_{i=0}^{n-1}\frac{\lambda^i}{i!}x^{i-1}\mathrm{e}^{-\lambda x}(i-\lambda x) \\
&= -\mathrm{e}^{-\lambda x}\Big(\sum_{i=0}^{n-1}\frac{\lambda^i}{i!}x^{i-1}i - \sum_{i=0}^{n-1}\frac{\lambda^i}{i!}x^{i-1}\lambda x\Big) \\
&= -\mathrm{e}^{-\lambda x}\Big(\sum_{i=1}^{n-1}\frac{\lambda^i}{(i-1)!}x^{i-1} - \sum_{i=0}^{n-1}\frac{\lambda^{i+1}}{i!}x^i\Big) \\
&= -\mathrm{e}^{-\lambda x}\Big(\sum_{j=0}^{n-2}\frac{\lambda^{j+1}}{j!}x^j - \sum_{i=0}^{n-1}\frac{\lambda^{i+1}}{i!}x^i\Big) \\
&= -\mathrm{e}^{-\lambda x}\Big(-\frac{\lambda^n}{(n-1)!}x^{n-1}\Big) = \lambda\mathrm{e}^{-\lambda x}\frac{(\lambda x)^{n-1}}{(n-1)!},
\end{aligned}
$$

where in line 5 we used the product rule to compute $\frac{\mathrm{d}}{\mathrm{d}x}x^i\mathrm{e}^{-\lambda x}$ and in line 8 we set $j = i-1$.

Exercise 7. Patients arrive to a hospital according to a Poisson process $\{N(t),\ t \ge 0\}$ with parameter λ patients per hour. The doctor starts to examine patients only when the third patient arrives. Compute the probability that after 1.5 hours, the doctor has not started examining patients.

Answer. Let X_1, X_2, X_3 be independent random variables that are all exponentially distributed with parameter λ. We wish to find the probability that in $[0, 1.5]$ less than three patients arrive. We will use the relation between a Poisson process and exponentially distributed random variables: $\mathbb{P}(N(t) \le k) = \mathbb{P}(Y_1 + Y_2 + \cdots + Y_{k+1} > t)$ where Y_i are exponential random variables with the same parameter as the Poisson process $N(t)$, see Section 2.4 of the lecture notes for more details.

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\begin{aligned}
\mathbb{P}(\text{Less than three patients arrive in [0,1.5]}) &= \mathbb{P}(N(1.5) \le 2) \\
&= \mathbb{P}(X_1 + X_2 + X_3 > 1.5) = \sum_{k=0}^{2}\frac{(1.5\lambda)^k}{k!}\mathrm{e}^{-1.5\lambda}.
\end{aligned}
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