INSTRUCTION WEEK 6, 18 MARCH 2016

Exercise 1. A shopkeeper keeps track of the amount of money X he earns per day. The shopkeeper reports an average earning of $\mathbb{E}[X] = 1500$ per day. He also tells us that $\mathbb{E}[X^2] = 2.41 \cdot 10^6$. What type of approximating distribution would you fit for the distribution of X and what are its parameters?

Answer. We have $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 2.41 \cdot 10^6 - 1500^2 = 1.6 \cdot 10^5$, $\sigma(X) = \sqrt{\mathrm{Var}(X)} = 400$ and $c_X = \sigma(X)/\mathbb{E}[X] = 4/15$. Since $c_X < 1$ we fit an $E_{k-1,k}$ distribution, for which we require $1/k \leq c_X^2 \leq 1/(k-1)$ and thus $k = 15$. Then the approximating distribution is with probability p (resp. $1-p$) the sum of $k-1$ (resp. k) independent exponentials with common mean $1/\mu$. We choose, see page 13 of the lecture notes,

$$p = \frac{1}{1+c_X^2}\left(kc_X^2 - (k(1+c_X^2) - k^2c_X^2)^{1/2}\right) \approx 0.7548,$$
$$\mu = \frac{k-p}{\mathbb{E}[X]} \approx 0.0095.$$

Exercise 2. Verify Example 6.2.1 of the lecture notes.

Answer. As a time unit we choose 1 second. We have $\lambda = 1/3$, $\lambda_1 = 0.7\lambda$, $\lambda_2 = 0.2\lambda$, $\lambda_3 = 0.1\lambda$, $\mu_1 = 1$, $\mu_2 = 1/4$, $\mu_3 = 1/10$ and $\rho_1 = 7/30$, $\rho_2 = 4/15$, $\rho_3 = 1/3$. The overall mean service time is $\mathbb{E}[B] = \sum_{i=1}^3 \lambda_i/(\lambda\mu_i) = 2.5$ and thus $\rho = \lambda\mathbb{E}[B] = 5/6$. Note that the service requirement distribution is a hyperexponential distribution. So, $\mathbb{E}[B^2] = 2\sum_{i=1}^3 \lambda_i/(\lambda\mu_i^2) = 27.8$. The expected residual service time is $\mathbb{E}[R] = \mathbb{E}[B^2]/(2\mathbb{E}[B]) = 5.56$ and $\mathbb{E}[W]$ follows from equation (5.1) of the lecture notes.
For the non-preemptive priorities, the expected waiting time per type follows from, see equation (6.4) of the lecture notes,

$$\mathbb{E}[W_i] = \frac{\rho\mathbb{E}[R]}{(1 - \sum_{j=1}^{i}\rho_j)(1 - \sum_{j=1}^{i-1}\rho_j)}.$$

Exercise 3. Consider a single processor which handles three types of tasks. Type-i tasks arrive according to a Poisson process with rate λ_i and the service requirements of type-i tasks are exponentially distributed with mean $1/\mu_i$, $i = 1,2,3$, with $\rho = \lambda_1/\mu_1 + \lambda_2/\mu_2 + \lambda_3/\mu_3 < 1$. Tasks are served in order of arrival.

(a) Argue why the system may be viewed as an $M/H_3/1$ queue, where H_3 represents a hyperexponential service requirement distribution.

(b) Determine the mean waiting time of type-1 tasks. (**A**: $\mathbb{E}[W_1] = (1-\rho)^{-1}\sum_{i=1}^3 \rho_i/\mu_i$)

(c) Suppose that at some point in time there are at least three tasks present in the system. What is the probability that the first two waiting tasks (thus excluding the task in service) belong to different types?

(d) Now suppose that the tasks are served according to a preemptive-resume priority strategy, which assigns the highest priority to type-1 tasks, the next highest priority to type-2 tasks, and the lowest priority to type-3 tasks. Determine the mean waiting time of a type-i task.

Answer. Define $\lambda = \lambda_1 + \lambda_2 + \lambda_3$.

(a) With probability λ_i/λ, the service time distribution B is exponentially distributed with parameter μ_i.

(b) Note that the mean waiting time for type-1 tasks actually do no depend on the type, i.e. $\mathbb{E}[W_1] = \mathbb{E}[W_2] = \mathbb{E}[W_3]$. The first and second moment of the service time distribution are $\mathbb{E}[B] = \sum_{i=1}^3 \lambda_i/(\lambda\mu_i)$ and $\mathbb{E}[B^2] = 2\sum_{i=1}^3 \lambda_i/(\lambda\mu_i^2)$. The expected residual service time is $\mathbb{E}[R] = \mathbb{E}[B^2]/(2\mathbb{E}[B]) = \sum_{i=1}^3 \rho_i/(\rho\mu_i)$. We finally obtain from equation (5.1) of the lecture notes $\mathbb{E}[W_1] = \rho\mathbb{E}[R]/(1-\rho) = (1-\rho)^{-1}\sum_{i=1}^3 \rho_i/\mu_i$.

(c) Let F be the type of the first waiting task and S be the type of the second waiting task. Then we find the requested probability as $\sum_{i=1}^3 \mathbb{P}(S \neq i | F = i)\mathbb{P}(F = i) = \sum_{i=1}^3 (1 - \lambda_i/\lambda)\lambda_i/\lambda$.

(d) The expected waiting time per task can be obtained with the help of the equation on page 48 of the lectures notes, namely

$$\mathbb{E}[W_i] = \frac{\sum_{j=1}^i \rho_j \mathbb{E}[R_j]}{(1 - \sum_{j=1}^i \rho_j)(1 - \sum_{j=1}^{i-1} \rho_j)},$$

with $\mathbb{E}[R_i] = 1/\mu_i$ because of the memoryless property.

Exercise 4. A machine mounts electronic components on three different types of printed circuit boards, type A, B and C boards say. On average 54 type-A boards arrive per hour, 48 type-B boards, and 18 type-C boards. The arrival processes are Poisson. The mounting times are exactly 20 seconds for type A, 30 seconds for type B, and 40 seconds for type C. The boards are processed in order of arrival.

(a) Calculate the mean overall waiting time $\mathbb{E}[W]$. (**A**: $\mathbb{E}[W] = 13/6$)

(b) Calculate for each type of printed circuit board the mean sojourn time and also calculate the mean overall sojourn time. (**A**: $\mathbb{E}[S_A] = 15/6$, $\mathbb{E}[S_B] = 16/6$, $\mathbb{E}[S_C] = 17/6$, $\mathbb{E}[S] = 157/60$)

From now on, suppose that there are priority rules in effect. Type-A boards have the highest priority, type-B boards have intermediate priority, and type-C boards have the lowest priority. The priorities are non-preemptive.

(c) Compute $\mathbb{E}[W_A]$ and $\mathbb{E}[W_B]$, the mean waiting times of type-A and type-B boards, respectively. (**A**: $\mathbb{E}[W_A] = 13/42$, $\mathbb{E}[W_B] = 65/63$)

(d) Use the work conservation law in combination with your answers to questions (a) and (c) to calculate $\mathbb{E}[W_C]$, the mean waiting time of type-C boards. (**A**: $\mathbb{E}[W_C] = 65/9$)

(e) Calculate for each type of printed circuit board the mean sojourn time. (**A**: $\mathbb{E}[S_A] = 27/42$, $\mathbb{E}[S_B] = 193/126$, $\mathbb{E}[S_C] = 71/9$)

Answer. As a time unit we choose 1 minute. We have the following parameter values.

i	λ_i	$\mathbb{E}[B_i]$	$\mathbb{E}[B_i^2]$	$\mathbb{E}[R_i]$	ρ_i
A	9/10	1/3	1/9	1/6	3/10
B	4/5	1/2	1/4	1/4	2/5
C	3/10	2/3	4/9	1/3	1/5

Finally, the aggregated arrival rate is $\lambda = \lambda_A + \lambda_B + \lambda_C = 2$, the load $\rho = \rho_A + \rho_B + \rho_C = 9/10$ and the residual service time $\mathbb{E}[R] = \sum_{i \in \{A,B,C\}} \rho_i/\rho \mathbb{E}[R_i] = 13/54$.

(a) We have $\mathbb{E}[W_A] = \mathbb{E}[W_B] = \mathbb{E}[W_C] = \mathbb{E}[W] = \rho \mathbb{E}[R]/(1-\rho) = 13/6$.

(b) The mean sojourn time per type follows from $\mathbb{E}[S_i] = \mathbb{E}[W] + \mathbb{E}[B_i],\ i \in \{A, B, C\}$ and we get $\mathbb{E}[S_A] = 15/6$, $\mathbb{E}[S_B] = 16/6$, $\mathbb{E}[S_C] = 17/6$. The mean overall sojourn time follows by weighing the sojourn times of each type: $\mathbb{E}[S] = \sum_{i \in \{A,B,C\}} \lambda_i/\lambda \mathbb{E}[S_i] = 157/60$.

(c) The mean waiting time per type is given by equation (6.4) of the lecture notes. We get $\mathbb{E}[W_A] = 13/42$ and $\mathbb{E}[W_B] = 65/63$.

(d) Use the conservation law $E[W] = \sum_{i \in \{A,B,C\}} \rho_i/\rho \mathbb{E}[W_i]$ to obtain $\mathbb{E}[W_C] = 65/9$.

(e) The mean sojourn time per type follows from $\mathbb{E}[S_i] = \mathbb{E}[W_i] + \mathbb{E}[B_i]$. We obtain $\mathbb{E}[S_A] = 27/42$, $\mathbb{E}[S_B] = 193/126$, $\mathbb{E}[S_C] = 71/9$.