Instruction week 5, 11 March 2016

Exercise 1. Consider an $M/M/1$ queue with arrival rate λ and service rate μ, with $\mu > \lambda$. Let $\mathbb{E}[C_n]$ be the expected time for the system to empty, starting with n customers, $n = 0, 1, \dots$ (so $C_0 = 0$).

(a) Show that the $\mathbb{E}[C_n]$'s satisfy the following recursive relationship

$$\mathbb{E}[C_n] = \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}\mathbb{E}[C_{n-1}] + \frac{\lambda}{\lambda+\mu}\mathbb{E}[C_{n+1}], \quad n = 1, 2, \dots \tag{1}$$

Hint. Draw the transition rate diagram.

(b) Show that $\mathbb{E}[C_1] = 1/(\mu - \lambda)$.

Hint. Argue that $\mathbb{E}[C_1]$ is the expected time that the server is working without interruption (busy period) and that $1/\lambda$ is the expected time that the system is empty without interruption (idle period); then $\mathbb{E}[C_1]/(\mathbb{E}[C_1] + 1/\lambda)$ must be equal to λ/μ.

(c) Argue that the expected time to decrease the queue length from 2 customers to 1 customer is equal to $\mathbb{E}[C_1]$, so that $\mathbb{E}[C_2] = 2\mathbb{E}[C_1]$ and in general $\mathbb{E}[C_n] = n\mathbb{E}[C_1]$. Verify this by substitution into the recursive relation (1).

Exercise 2. One is planning to build new telephone boxes near the railway station. Measurements showed that 120 persons per hour want to make a phone call. These persons arrive to the telephone boxes according to a Poisson process. The duration of a call is exponentially distributed with a mean of 1 minute. What is the minimum number of telephone boxes such that

(a) The probability that a person has to wait is less than 6%? (**A**: 5)

(b) The mean waiting time is less than 0.1 minutes? (**A**: 4)

(c) At most 5% has to wait longer than 2 minutes? (**A**: 4)

Exercise 3. Consider an $M/E_2/1$ queueing system with arrival rate λ and a service consisting of two exponential phases, both with mean $1/\mu$. So, the service time is Erlang-2(μ) distributed. We wish to compute the queue length distribution using the generating function approach as described in Section 5.5.

(a) First, we need to compute the generating function of the number of arrivals during the service of the n-th customer. This generating function is given by

$$P_A(z) = \sum_{i=0}^{\infty} \mathbb{P}(A = i) z^i = \int_0^{\infty} \mathrm{e}^{-\lambda t(1-z)} f_B(t)\,\mathrm{d}t,$$

where $f_B(t)$ is the density of the service time distribution. Show that

$$P_A(z) = \left(\frac{\mu}{\mu + \lambda(1-z)}\right)^2.$$

(b) We can now compute the generating function of the number of customers that is left behind by a departing customer, i.e. $P_{X_d}(z)$. It is given by

$$P_{X_d}(z) = \sum_{i=0}^{\infty} \mathbb{P}(X_d = i) z^i = \frac{(1-\rho)(1-z)P_A(z)}{P_A(z) - z}.$$

Show that, assuming $z \neq 1$ and $\rho = \lambda \mathbb{E}[B] = 2\lambda/\mu$,

$$P_{X_d}(z) = \frac{(1-\rho)\mu^2}{\mu^2 - 2\lambda\mu z - \lambda^2 z(1-z)} = \frac{1-\rho}{1 - \rho z - \rho^2 z(1-z)/4}.$$

(c) Assume $\rho = 1/3$, then show that

$$P_{X_d}(z) = \frac{24/5}{4-z} - \frac{24/5}{9-z} = \frac{6}{5}\frac{1}{1-z/4} - \frac{8}{15}\frac{1}{1-z/9}.$$

This actually indicates that the queue length distribution is a mixture of two geometric distributions, namely

$$p_n = \frac{6}{5}\left(\frac{1}{4}\right)^n - \frac{8}{15}\left(\frac{1}{9}\right)^n.$$