Stochastic performance modelling (2WB60)

Instruction week 2, 19 February 2016

Exercise 1. Consider a Markov chain with state space $\{1, 2, 3\}$ and transition probabilities $p_{12} = 1$, $p_{21} = 1/2$, $p_{23} = 1/2$, $p_{32} = 1/2$ and $p_{33} = 1/2$. Determine the steady-state distribution of this Markov chain. (**A**: $\pi = (1, 2, 2)/5$)

Exercise 2. A processor is inspected weekly in order to determine its condition. The condition of the processor can either be *perfect*, *good*, *reasonable*, or *bad*. A new processor is still perfect after one week with probability 0.7, with probability 0.2 the state is good, and with probability 0.1 it is reasonable. A processor in good condition is still good after one week with probability 0.6, reasonable with probability 0.2, and bad with probability 0.2. A processor in reasonable condition is still reasonable after one week with probability 0.5 and bad with probability 0.5. A bad processor must be repaired. The repair takes one week, after which the processor is again in perfect condition.

(a) Formulate a Markov chain that describes the state of the machine, and draw the corresponding transition diagram.

(b) Determine the steady-state distribution of the Markov chain. (**A**: $\pi = (10, 5, 4, 3)/22$)

Exercise 3. Consider the Markov chain described in Example 3.3.

(a) Derive the balance equations

$$\pi_i = \pi_{i+1}p + \pi_{i-1}q, \quad i \geq 1, \tag{1}$$

and

$$\pi_0 = \pi_0 p + \pi_1 p, \tag{2}$$

by using the fact that there is a balance between the probability of moving to state i and the probability of moving away from state i ('in = out').

(b) Show that the limiting distribution satisfies $\pi_i = (q/p)^i(1 - q/p),\ i \geq 0$.

(c) Also derive the limiting distribution using probability generating functions.

Hint. Let L be the number of cells in the buffer. The probability generating function of L is given by

$$P_L(z) = \mathbb{E}[z^L] = \sum_{k=0}^{\infty} \mathbb{P}(L = k)z^k = \sum_{k=0}^{\infty} \pi_k z^k. \tag{3}$$

Think about multiplying the equations by a certain factor and then summing over all equations. The probability generating function of a geometric random variable X with parameter p is given by

$$P_X(z) = \mathbb{E}[z^X] = \sum_{k=0}^{\infty} \mathbb{P}(X = k)z^k = \frac{1-p}{1-pz}. \tag{4}$$