Stochastic performance modelling (2WB60)

Instruction week 5, 11 March 2016

Exercise 1. Consider an $M/M/1$ queue with arrival rate λ and service rate μ, with $\mu > \lambda$. Let $\mathbb{E}[C_n]$ be the expected time for the system to empty, starting with n customers, $n = 0, 1, \ldots$ (so $C_0 = 0$).

(a) Show that the $\mathbb{E}[C_n]$'s satisfy the following recursive relationship

$$\mathbb{E}[C_n] = \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}\mathbb{E}[C_{n-1}] + \frac{\lambda}{\lambda+\mu}\mathbb{E}[C_{n+1}], \quad n = 1, 2, \ldots \tag{1}$$

Hint. Draw the transition rate diagram.

(b) Show that $\mathbb{E}[C_1] = 1/(\mu - \lambda)$.

Hint. Argue that $\mathbb{E}[C_1]$ is the expected time that the server is working without interruption (busy period) and that $1/\lambda$ is the expected time that the system is empty without interruption (idle period); then $\mathbb{E}[C_1]/(\mathbb{E}[C_1] + 1/\lambda)$ must be equal to λ/μ.

(c) Argue that the expected time to decrease the queue length from 2 customers to 1 customer is equal to $\mathbb{E}[C_1]$, so that $\mathbb{E}[C_2] = 2\mathbb{E}[C_1]$ and in general $\mathbb{E}[C_n] = n\mathbb{E}[C_1]$. Verify this by substitution into the recursive relation (1).

Answer.

(a) The expected time we stay in state n of the Markov process is $1/(\lambda + \mu)$. We then make a jump to state $n-1$ with probability $\mu/(\lambda+\mu)$, where we have an expected time for the system to empty, starting with $n-1$ customers, and similarly for jumping to state $n+1$.

(b) Actually, $\mathbb{E}[C_1]$ is just the expected duration of a busy period. The reasoning then follows from method 1 of Section 5.4.

(c) When determining the length of a busy period, the order in which we serve the customers does not matter (for this model). So, when we start with 2 customers, we can put aside one of the two customers, say the second customer, and start a busy period with the first customer. The second customer is only allowed to start service when the busy period generated by the first customer is finished. The expected duration of the busy period generated by the first customer is $\mathbb{E}[C_1]$, which is then immediately followed by another busy period started by the second customer, which again has length $\mathbb{E}[C_1]$. Thus $\mathbb{E}[C_2] = 2\mathbb{E}[C_1]$. This reasoning extends to $\mathbb{E}[C_n]$.

Exercise 2. One is planning to build new telephone boxes near the railway station. Measurements showed that 120 persons per hour want to make a phone call. These persons arrive to the telephone boxes according to a Poisson process. The duration of a call is exponentially distributed with a mean of 1 minute. What is the minimum number of telephone boxes such that

(a) The probability that a person has to wait is less than 6%? (**A**: 5)

(b) The mean waiting time is less than 0.1 minutes? (**A**: 4)

(c) At most 5% has to wait longer than 2 minutes? (**A**: 4)

Answer. As a time unit we choose 1 minute. Then, the arrival rate is $\lambda = 2$, the service rate is $\mu = 1$ and the offered load is $\rho := \lambda/(c\mu)$, where c is the number of telephone boxes. We compute the equilibrium probabilities as explained in Section 4.4.1. Namely,

$$\begin{aligned} p_n &= \frac{(c\rho)^n}{n!} p_0, \quad n = 0, 1, \dots, c, \\ p_{c+n} &= \rho^n p_c, \quad n = 0, 1, \dots, \\ p_0 &= \Big(\sum_{n=0}^{c-1} \frac{(c\rho)^n}{n!} + \frac{(c\rho)^c}{c!} \frac{1}{1-\rho} \Big)^{-1}. \end{aligned}$$

Note that we need at least $c = 3$ telephone boxes for the system to be stable. The results are summarized in the table below.

(a) We compute the probability to wait $\Pi_W = p_c/(1-\rho)$ and find that we need a minimum of $c = 5$ telephone boxes.

(b) We compute the mean waiting time $\mathbb{E}[W] = \Pi_W/((1-\rho)c\mu)$ and find that we need a minimum of $c = 4$ servers.

(c) The distribution of the waiting time is given by $\mathbb{P}(W > t) = \Pi_W \mathrm{e}^{-c\mu(1-\rho)t}$, $t \geq 0$. For $t = 2$, we find that we need $c = 4$ telephone boxes.

c	3	4	5
Π_W	0.4444	0.1739	0.0597
$\mathbb{E}[W]$	0.4444	0.0870	0.0199
$\mathbb{P}(W > 2)$	0.0601	0.0032	0.0001

Exercise 3. Consider an $M/E_2/1$ queueing system with arrival rate λ and a service consisting of two exponential phases, both with mean $1/\mu$. So, the service time is Erlang-2(μ) distributed. We wish to compute the queue length distribution using the generating function approach as described in Section 5.5.

(a) First, we need to compute the generating function of the number of arrivals during the service of the n-th customer. This generating function is given by

$$P_A(z) = \sum_{i=0}^{\infty} \mathbb{P}(A = i) z^i = \int_0^{\infty} \mathrm{e}^{-\lambda t(1-z)} f_B(t)\, \mathrm{d}t,$$

where $f_B(t)$ is the density of the service time distribution. Show that

$$P_A(z) = \Big(\frac{\mu}{\mu + \lambda(1-z)} \Big)^2.$$

(b) We can now compute the generating function of the number of customers that is left behind by a departing customer, i.e. $P_{X_d}(z)$. It is given by

$$P_{X_d}(z) = \sum_{i=0}^{\infty} \mathbb{P}(X_d = i) z^i = \frac{(1-\rho)(1-z)P_A(z)}{P_A(z) - z}.$$

Show that, assuming $z \neq 1$ and $\rho = \lambda \mathbb{E}[B] = 2\lambda/\mu$,

$$P_{X_d}(z) = \frac{(1-\rho)\mu^2}{\mu^2 - 2\lambda\mu z - \lambda^2 z(1-z)} = \frac{1-\rho}{1 - \rho z - \rho^2 z(1-z)/4}.$$

(c) Assume $\rho = 1/3$, then show that

$$P_{X_d}(z) = \frac{24/5}{4-z} - \frac{24/5}{9-z} = \frac{6}{5}\frac{1}{1-z/4} - \frac{8}{15}\frac{1}{1-z/9}.$$

This actually indicates that the queue length distribution is a mixture of two geometric distributions, namely

$$p_n = \frac{6}{5}\left(\frac{1}{4}\right)^n - \frac{8}{15}\left(\frac{1}{9}\right)^n.$$

Answer.

(a) The density of the service time distribution is $f_B(t) = \mu^2 t\mathrm{e}^{-\mu t}$. Thus, the integral becomes

$$P_A(z) = \int_0^\infty \mathrm{e}^{-\lambda t(1-z)}\mu^2 t\,\mathrm{e}^{-\mu t}\,\mathrm{d}t = \mu^2\int_0^\infty t\,\mathrm{e}^{-(\mu+\lambda(1-z))t}\,\mathrm{d}t = \left(\frac{\mu}{\mu+\lambda(1-z)}\right)^2.$$

(b) Substitute the above expression for $P_A(z)$ to obtain

$$\begin{aligned}
P_{X_d}(z) &= \frac{(1-\rho)(1-z)\left(\frac{\mu}{\mu+\lambda(1-z)}\right)^2}{\left(\frac{\mu}{\mu+\lambda(1-z)}\right)^2 - z} \\
&= \frac{(1-\rho)(1-z)\mu^2}{\mu^2 - z(\mu+\lambda(1-z))^2} \\
&= \frac{(1-\rho)(1-z)\mu^2}{\mu^2 - z(\mu^2 + 2\lambda\mu(1-z) + \lambda^2(1-z)^2)} \\
&= \frac{(1-\rho)\mu^2}{\mu^2 - 2\lambda\mu z - \lambda^2 z(1-z)} \\
&= \frac{1-\rho}{1-\rho z - \rho^2 z(1-z)/4}.
\end{aligned}$$

(c) Substitute $\rho = 1/3$ to get

$$\begin{aligned}
P_{X_d}(z) &= \frac{2/3}{1-z/3-z(1-z)/36} = \frac{24}{36-13z+z^2} = \frac{24}{(4-z)(9-z)} \\
&= \frac{24/5}{4-z} - \frac{24/5}{9-z} = \frac{6}{5}\frac{1}{1-z/4} - \frac{8}{15}\frac{1}{1-z/9}.
\end{aligned}$$