Stochastic performance modelling (2WB60)

Instruction week 3, 26 February 2016

Exercise 1. Consider the following Markov chains and verify for each Markov chain if it is irreducible, aperiodic and positive recurrent. See Condition 3.1 and its discussion in the lecture notes.

(a) A Markov chain with state space $\{1, 2\}$ and transition probability matrix given by

$$P = \begin{pmatrix} \epsilon & 1-\epsilon \\ 1-\epsilon & \epsilon \end{pmatrix}, \tag{1}$$

with $\epsilon \in [0, 1]$.

(b) A Markov chain with state space $\{1, 2, 3, 4, 5\}$ and transition probability matrix given by

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{2}$$

(c) A Markov chain with state space $\{1, 2, 3, 4\}$ and transition probability matrix given by

$$P = \begin{pmatrix} 1/2 & 1/3 & 0 & 1/6 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

Hint. Draw the transition diagrams.

Answer.

(a) The Markov chain is irreducible, aperiodic and positive recurrent for $\epsilon \in (0, 1)$. For $\epsilon = 0$, the Markov chain is irreducible, periodic with period 2 and positive recurrent. For $\epsilon = 1$, the Markov chain is reducible into two classes, namely $\{1\}$ and $\{2\}$. Both classes are positive recurrent (absorbing states) and aperiodic.

(b) The Markov chain is irreducible, periodic with period $d = 3$ and positive recurrent.

(c) The Markov chain is reducible into three classes, namely $\{1, 2\}$, $\{3\}$ and $\{4\}$, where the first class is transient and the last two classes are positive recurrent (absorbing states). Classes $\{3\}$ and $\{4\}$ are both aperiodic.

Exercise 2. Consider the Markov chain of exercise 1(b). Starting from state 3, how long does it take on average until the Markov chain reaches state 5 for the first time? (**A**: 8)
Hint. Use Remark 3.2.

Answer. Let $a_i(5)$ be the expected time it takes to reach state 5, starting from state i. Note that we are interested in $a_3(5)$. By a first-step analysis we get the following system of equations

$$\begin{aligned}
a_1(5) &= 1 + a_3(5),\\
a_2(5) &= 1 + a_1(5),\\
a_3(5) &= 1 + \frac{2}{3}a_2(5) + \frac{1}{3}a_4(5),\\
a_4(5) &= 1 + a_5(5),\\
a_5(5) &= 0.
\end{aligned}$$

Solving this system gives $a_3(5) = 8$.

Exercise 3. Consider a Markov process with state space $\{1,2,3\}$ and infinitesimal generator

$$Q = \begin{pmatrix} -1 & 1/2 & 1/2 \\ 1/2 & -1 & 1/2 \\ 3/2 & 3/2 & -3 \end{pmatrix} \tag{4}$$

(a) Determine the steady-state distribution of this Markov process. (**A**: $p = (3/7, 3/7, 1/7)$) *Hint.* Use Remark 3.3 and 3.4.

(b) Also determine the steady-state distribution of the underlying Markov chain. (**A**: $\pi = (1/3, 1/3, 1/3)$)

Answer.

(a) We use $0 = pQ$ with $p = (p_1, p_2, p_3)$ a row vector. This gives us the balance equations

$$\begin{aligned}
0 &= -p_1 + \frac{1}{2}p_2 + \frac{3}{2}p_3,\\
0 &= \frac{1}{2}p_1 - p_2 + \frac{3}{2}p_3,\\
0 &= \frac{3}{2}p_1 + \frac{3}{2}p_2 - 3p_3,
\end{aligned}$$

and the normalization condition $p_1 + p_2 + p_3 = 1$. This gives $p = (3/7, 3/7, 1/7)$.

(b) Use the fact that the transition rates are defined as $q_{ij} := \nu_i p_{ij},\ j \neq i$ and $q_{ii} := -\nu_i$ to obtain the transition probabilities p_{ij} of the underlying Markov chain at jump epochs. We have $p_{ij} = 1/2,\ j \neq i$ and thus the equilibrium distribution of the Markov chain $\pi = (\pi_1, \pi_2, \pi_3)$ satisfying $\pi = \pi P$ and $\pi_1 + \pi_2 + \pi_3 = 1$ is given by $\pi = (1/3, 1/3, 1/3)$.

Exercise 4. Consider a Markov process with state space $\{0,1,2\}$ and infinitesimal generator

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & \mu & -\mu \end{pmatrix}. \tag{5}$$

(a) Derive the parameters ν_i and p_{ij} for this Markov process.

(b) Determine the expected time to go from state 1 to state 0. (**A**: $\frac{1+\frac{\lambda}{\mu}}{\mu}$)

Answer.

(a) Use the definition of $q_{ij},\ j \neq i$ and q_{ii} to obtain

$$\nu_0 = \lambda, \quad \nu_1 = \lambda + \mu, \quad \nu_2 = \mu,$$

and

$$p_{01} = 1, \quad p_{10} = 1 - p_{12} = \frac{\mu}{\lambda + \mu}, \quad p_{21} = 1.$$

(b) Let $a_i(0)$ be the expected time it takes to reach state 0, starting from state i. Using a one-step analysis we obtain the following system of equations

$$\begin{aligned} a_0(0) &= 0, \\ a_1(0) &= \frac{1}{\nu_1} + p_{10}a_0(0) + p_{12}a_2(0), \\ a_2(0) &= \frac{1}{\nu_2} + p_{21}a_1(0), \end{aligned}$$

with solution

$$a_1(0) = \frac{\frac{1}{\nu_1} + p_{12}\frac{1}{\nu_2}}{1 - p_{12}p_{21}} = \frac{1 + \frac{\lambda}{\mu}}{\mu}.$$

Exercise 5. A repair man fixes broken TV sets. Broken TV sets arrive at his repair shop according to a Poisson process, with an average of 10 broken TV sets per work day (8 hours). The repair times are exponentially distributed with a mean of 30 minutes.

(a) What is the equilibrium probability of having k TV sets in the system? (**A**: $3/8(5/8)^k$)

(b) What is the fraction of time that the repair man has no work to do? (**A**: 3/8)

(c) How many TV sets are, on average, at his repair shop? (**A**: 5/3)

(d) What is the mean sojourn time (waiting time plus repair time) of a TV set? (**A**: 4/3)

Answer.

(a) We take as a time unit an hour. Then, we have the arrival rate $\lambda = 5/4$ TV sets per hour and service rate $\mu = 2$ TV sets per hour. The load on the system is thus $\rho = \lambda/\mu = 5/8$. The repair shop can be modeled as an $M/M/1$ queueing system. The equilibrium probability of having k TV sets in the system, i.e. p_k, is given by

$$p_k = (1 - \rho)\rho^k = 3/8(5/8)^k.$$

(b) $1 - \rho = 3/8$.

(c) $\mathbb{E}[L] = \frac{\rho}{1-\rho} = 5/3$.

(d) From Little's law we get $\mathbb{E}[S] = \mathbb{E}[L]/\lambda = \frac{\rho}{1-\rho}\frac{1}{\lambda} = \frac{1}{1-\rho}\frac{1}{\mu} = 4/3$.