Stochastic performance modelling (2WB60)

Instruction week 3, 26 February 2016

Exercise 1. Consider the following Markov chains and verify for each Markov chain if it is irreducible, aperiodic and positive recurrent. See Condition 3.1 and its discussion in the lecture notes.

(a) A Markov chain with state space $\{1, 2\}$ and transition probability matrix given by

$$P = \begin{pmatrix} \epsilon & 1-\epsilon \\ 1-\epsilon & \epsilon \end{pmatrix}, \tag{1}$$

with $\epsilon \in [0, 1]$.

(b) A Markov chain with state space $\{1, 2, 3, 4, 5\}$ and transition probability matrix given by

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{2}$$

(c) A Markov chain with state space $\{1, 2, 3, 4\}$ and transition probability matrix given by

$$P = \begin{pmatrix} 1/2 & 1/3 & 0 & 1/6 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

Hint. Draw the transition diagrams.

Exercise 2. Consider the Markov chain of exercise 1(b). Starting from state 3, how long does it take on average until the Markov chain reaches state 5 for the first time? (**A**: 8)
Hint. Use Remark 3.2.

Exercise 3. Consider a Markov process with state space $\{1, 2, 3\}$ and infinitesimal generator

$$Q = \begin{pmatrix} -1 & 1/2 & 1/2 \\ 1/2 & -1 & 1/2 \\ 3/2 & 3/2 & -3 \end{pmatrix} \tag{4}$$

(a) Determine the steady-state distribution of this Markov process. (**A**: $p = (3/7, 3/7, 1/7)$)
Hint. Use Remark 3.3 and 3.4.

(b) Also determine the steady-state distribution of the underlying Markov chain. (**A**: $\pi = (1/3, 1/3, 1/3)$)

Exercise 4. Consider a Markov process with state space $\{0, 1, 2\}$ and infinitesimal generator

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & \mu & -\mu \end{pmatrix}. \tag{5}$$

(a) Derive the parameters ν_i and p_{ij} for this Markov process.

(b) Determine the expected time to go from state 1 to state 0. (**A**: $\frac{1+\frac{\lambda}{\mu}}{\mu}$)

Exercise 5. A repair man fixes broken TV sets. Broken TV sets arrive at his repair shop according to a Poisson process, with an average of 10 broken TV sets per work day (8 hours). The repair times are exponentially distributed with a mean of 30 minutes.

(a) What is the equilibrium probability of having k TV sets in the system? (**A**: $3/8(5/8)^k$)

(b) What is the fraction of time that the repair man has no work to do? (**A**: 3/8)

(c) How many TV sets are, on average, at his repair shop? (**A**: 5/3)

(d) What is the mean sojourn time (waiting time plus repair time) of a TV set? (**A**: 4/3)