6 A system with production-to-stock

In this chapter we will look at a simple model for a system with production-to-stock. The advantage of production-to-stock in comparison with production-to-order is that we can improve the service of customers by reducing the waiting times. However, this is done by keeping inventory which, of course, leads to extra costs.

6.1 The model

Like in the $M/M/1$ system, product demand is modelled by a Poisson process with rate λ. Furthermore, the processing times of the products are exponentially distributed with parameter μ. As usual, we require $\rho = \lambda/\mu < 1$.

The manufacturing system uses the following *base-stock policy*: produce as soon as the number of products on stock is smaller than some base-stock level S. We assume that any demand, when out of stock, is backordered and is filled as soon as a product becomes available (i.e., *complete backordering*). The alternative case (*complete lost sales*), in which any demand, when out of stock, is lost (i.e., the customer goes elsewhere) will be briefly described in remark 6.1.

In order to be able to compare the advantage of reducing waiting times with the disadvantage of keeping inventory, we introduce two types of costs. On one hand, we have holding costs, h, per item per unit of time. On the other hand we have backordering costs, b, per item per unit of time. We are looking for the base-stock level, S, minimizing the mean costs per unit of time.

The state of the system is described by the number of items on stock minus the number of backorders. Hence, the set of possible states is given by $\{S, S-1, \dots, 1, 0, -1, \dots\}$. The positive states correspond to states with some items on stock and no backorders, the negative states correspond to states with some backorders and no items on stock. The flow diagram for this model is shown in figure 1.

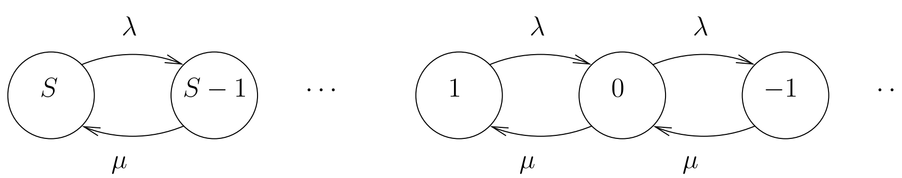

Figure 1: Flow diagram for the production-to-stock model

Let p_n be the equilibrium probability that the system is in state n. From the similarity between the flow diagram in figure 1 and the one for the $M/M/1$ model, we immediately see that p_n is given by

$$p_n = (1-\rho)\rho^{S-n}, \qquad n = S, S-1, S-2, \dots.$$

This distribution can be used to derive the equilibrium distribution for the number of items on stock, I, and the number of backorders, B:

$$\begin{aligned}
P(I=0) &= \sum_{n=0}^{\infty} p_{-n} = \rho^S, \\
P(I=n) &= p_n = (1-\rho)\rho^{S-n}, \qquad n = 1, 2, \ldots, S, \\
P(B=0) &= \sum_{n=0}^{S} p_n = 1 - \rho^{S+1}, \\
P(B=n) &= p_{-n} = (1-\rho)\rho^{S+n}, \qquad n = 1, 2, \ldots
\end{aligned}$$

From these distributions we obtain for the means

$$E(I) = S - \frac{\rho}{1-\rho}\left(1-\rho^S\right),$$

and

$$E(B) = \frac{\rho^{S+1}}{1-\rho}.$$

Hence the mean costs, $E(C)$, per unit of time equals

$$E(C) = E(I)\cdot h + E(B)\cdot b = \left(S - \frac{\rho}{1-\rho}\right)\cdot h + \frac{\rho^{S+1}}{1-\rho}\cdot(h+b).$$

Note that $E(C)$ is convex, so we can easily determine the value of S for which $E(C)$ is minimal by, e.g., by straightforward enumeration (starting with $S=0$, $S=1$ and so on).

Remark 6.1 The analysis of the model with lost-sales proceeds similarly; the main difference in the analysis is that the set of possible states in the lost-sales case is finite, namely $\{S, S-1, \ldots, 0\}$. Further, we now consider costs per item that is lost, instead of backorder costs per item per unit of time.

6.2 Setup costs

In some applications, not only holding costs and backordering costs play a role, but also setup costs, K, for every time a new production run is started. In such applications, the base-stock policy does not seem to be sensible, since this policy will probably lead to a lot of small production runs and, hence, to large setup costs. In that case, the (s, S) strategy seems reasonable: start a production run if the number of items on stock drops to level s and continue to produce as long as the number of items on stock is less than level S. Note that for $s = S-1$ this strategy coincides with the base-stock policy.

The state of the system under the (s, S) strategy can again be described by the number of items on stock minus the number of backorders. However, to describe the system *completely* we have to indicate, when the number of items on stock is greater than s, whether or not the machine is producing. In the sequel, we say that the system is in state n if there are n items on stock and the machine is producing, and the system is in state n^*

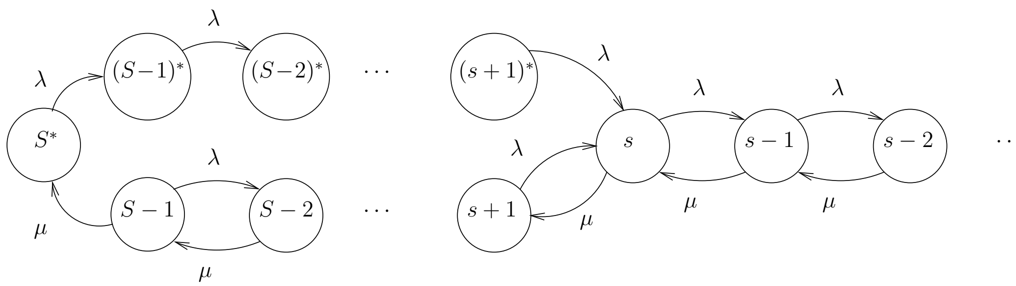

Figure 2: Flow diagram for the model with setup costs

if there are n items on stock and the machine is not producing. The set of possible states is now given by $\{S-1, S-2, \ldots, 1, 0, -1, \ldots\} \cup \{S^*, \ldots, (s+1)^*\}$. The flow diagram for this model is shown in figure 2.

Although the flow diagram does not have the nice chain structure as in the previous model, the equilibrium probabilities can again be found relatively easy. One may check that the solution is given by

$$\begin{aligned} p_{n^*} &= D, \qquad n = S, \ldots, s+1 \\ p_n &= D \sum_{j=1}^{S-n} \rho^j, \qquad n = S-1, \ldots, s+1 \\ p_n &= D\rho^{s-n} \sum_{j=1}^{S-s} \rho^j, \qquad n = s, s-1, \ldots . \end{aligned}$$

The constant D finally follows from the normalization equation. The mean costs per time unit is now equal to

$$E(C) = \lambda p_{(s+1)^*} K + \sum_{n=1}^{s} p_n nh + \sum_{n=s+1}^{S-1} (p_n + p_{n^*}) nh + p_{S^*} Sh + \sum_{n=1}^{\infty} p_{-n} nb.$$

The first term at the right-hand side corresponds to the mean set-up costs per time unit, the other terms to the mean holding and backordering costs. It is not so easy, however, to find the pair (s, S) minimizing $E(C)$.