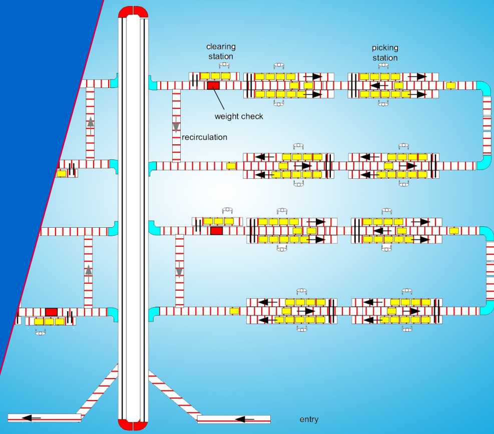
Stochastic Models of Manufacturing Systems

Ivo Adan



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Tuesday April 21

Organization

- 7 lectures (lecture of May 12 is canceled)
- Studyguide available (with notes, slides, assignments, references), see http://www.win.tue.nl/~iadan/4t400
- Examination consists of:
  - Weekly (8) take home assignments
  - Send in take home assignments individually
  - Best 7 out of 8 take home assignments count (40%)
  - Take home assignments of last week will be discussed in BZ
  - Final assignment, done in groups of two (60%)

Topics

- Basic probability (refresher)
- Basic statistics for discrete-event simulation
- Modeling and analysis of manufacturing systems:
  - Single-stage systems
  - Multi-stage flow lines
  - Job-shop systems
  - CONWIP systems

Modeling

Some basic steps:

- Identify the issues to be addressed
- Learn about the system
- Choose a modeling approach
- Develop and test the model
- Verify and validate the model
- Experiment with the model
- Present the results

Modeling

Various types of models:

- Physical models
- Simulation models
- Analytical models

But why modeling?

- Understanding
- Improvement
- Optimization
- Decision making

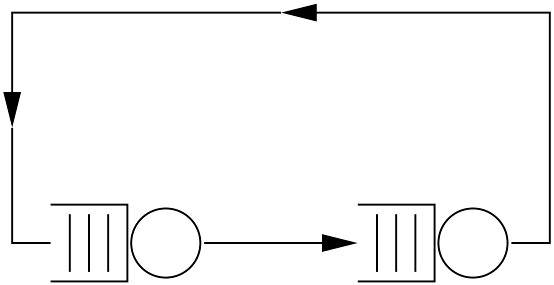
Modeling

Some issues:

- Complexity versus Simplicity
- Flexibility
- Data requirements
- Transparency

Analytical and simulation capability: Effective modeling requires both!

Some Improvements?

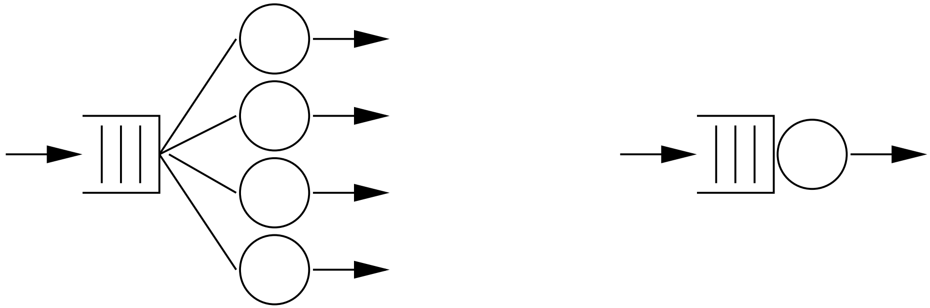


Servers are equally fast, 10 circulating jobs

Question: Replace one server by a server that is twice as fast.
How does this affect average throughput time? Throughtput?

Question: Does your answer change in case of more jobs? Less jobs?

Multiple Machines or a Single One?



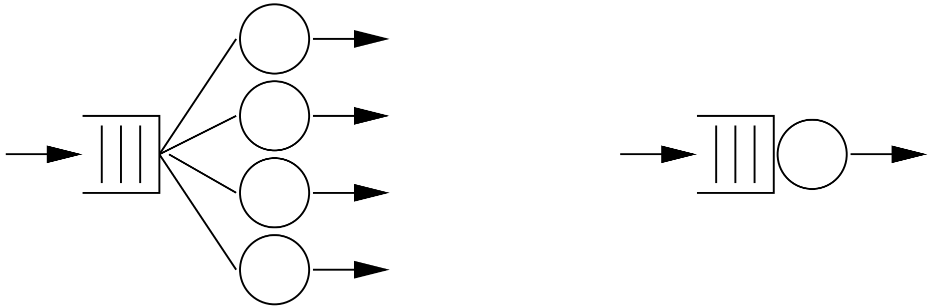
4 machines, or one machine that is four times faster?

Question: What do you prefer, 4 machines or one fast machine?

Question: What do you prefer if process time variability is high?

Question: What do you prefer if the load is low?

Multiple Machines or a Single One?

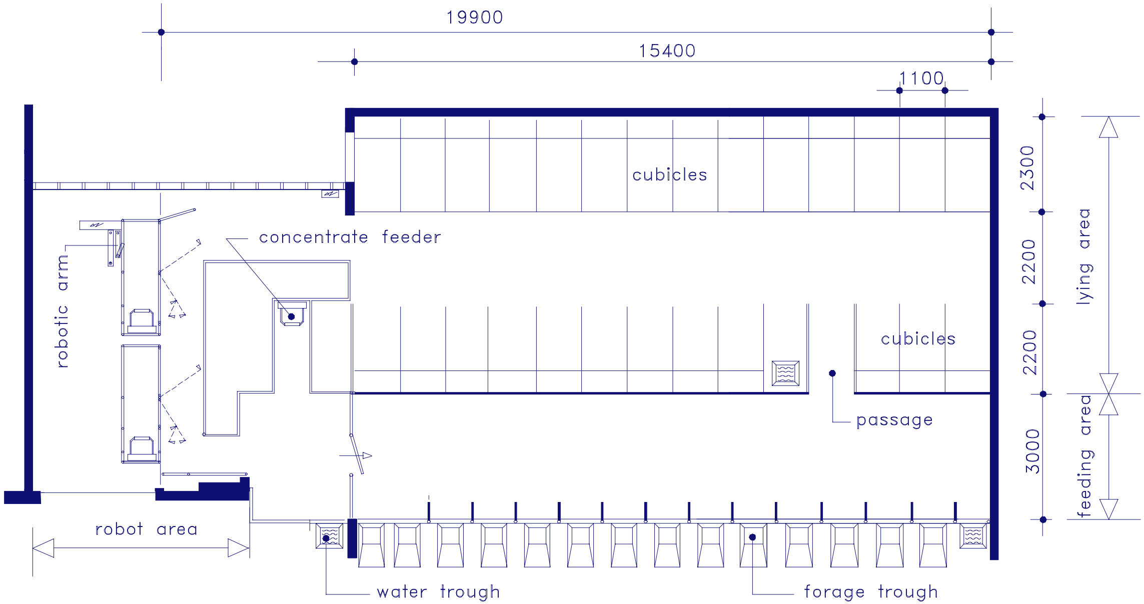


4 machines, or one machine that is four times faster?

The required information is captured in the following formula:

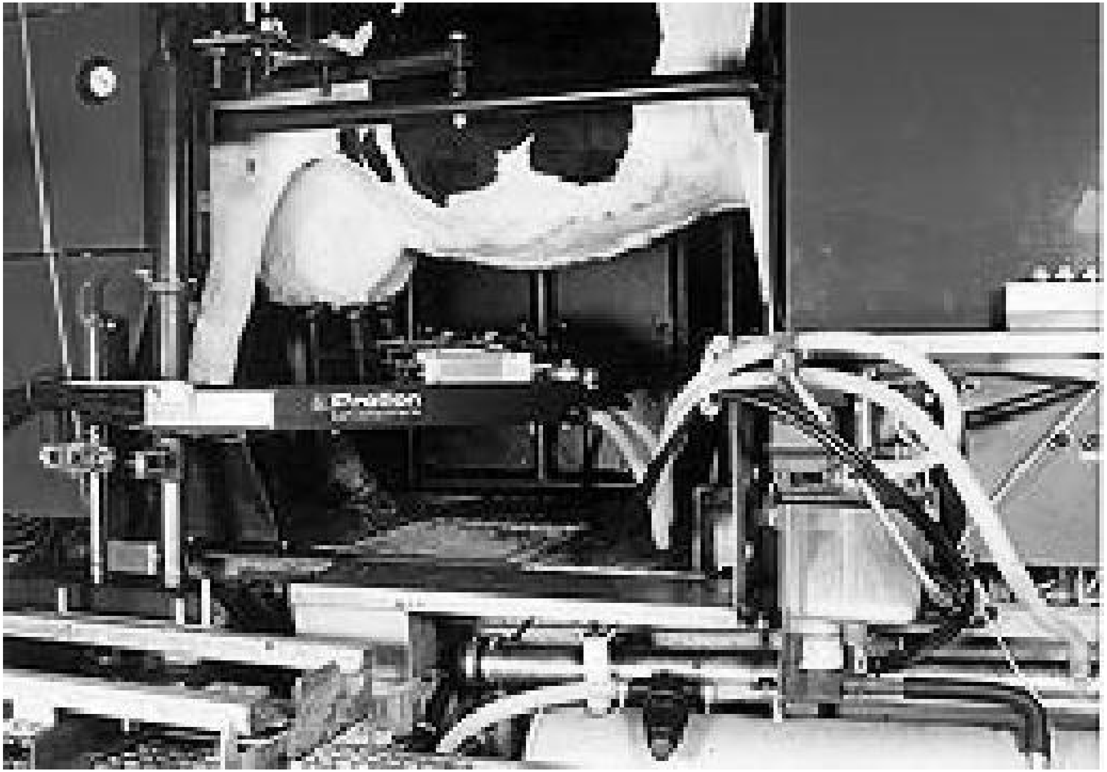
$$E(S) \approx \frac{\Pi_W}{1-\rho}\frac{E(R)}{c} + E(B)$$

Running example



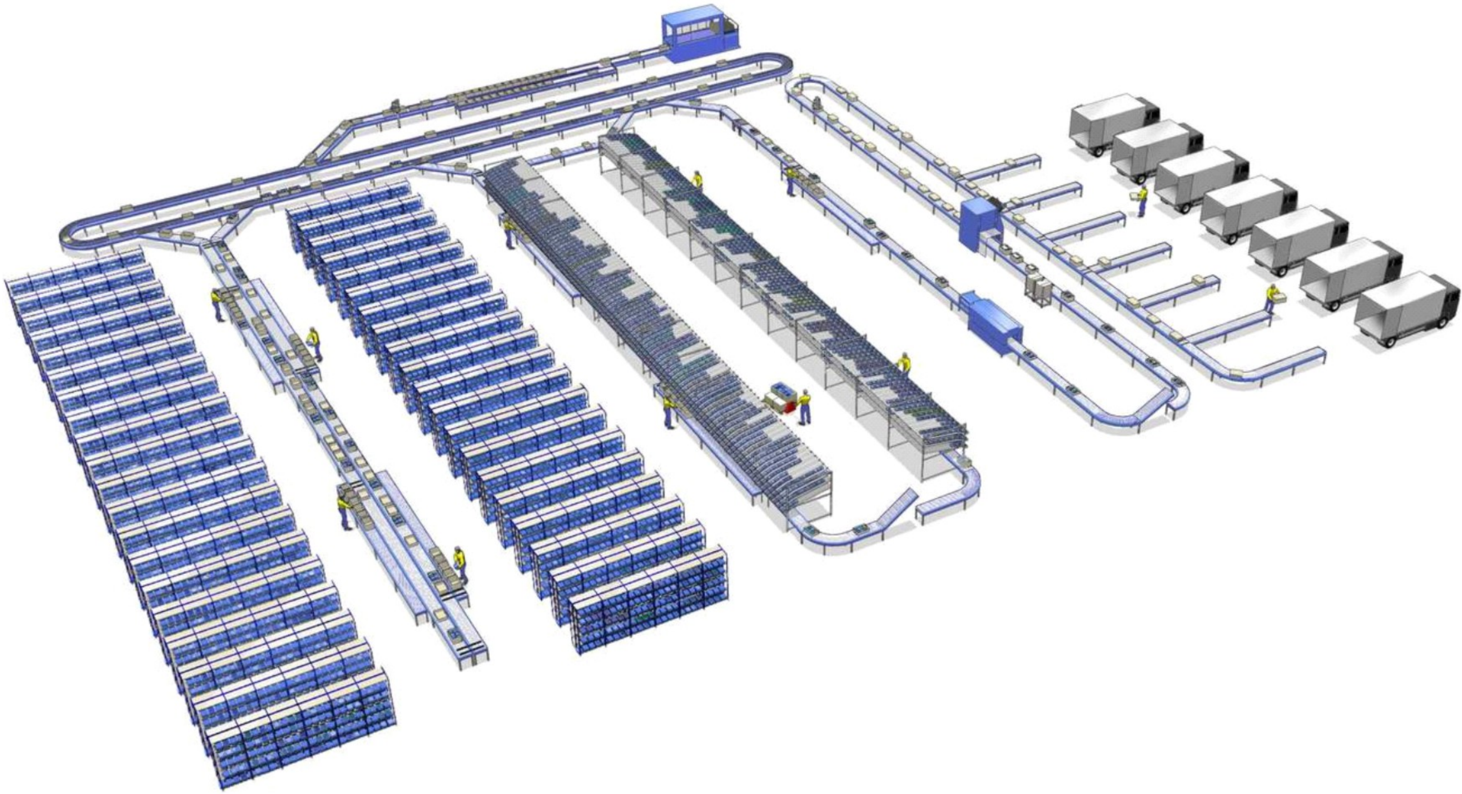
A robotic dairy barn

Milking robot



How to design such a barn?

Zone-Picking Systems



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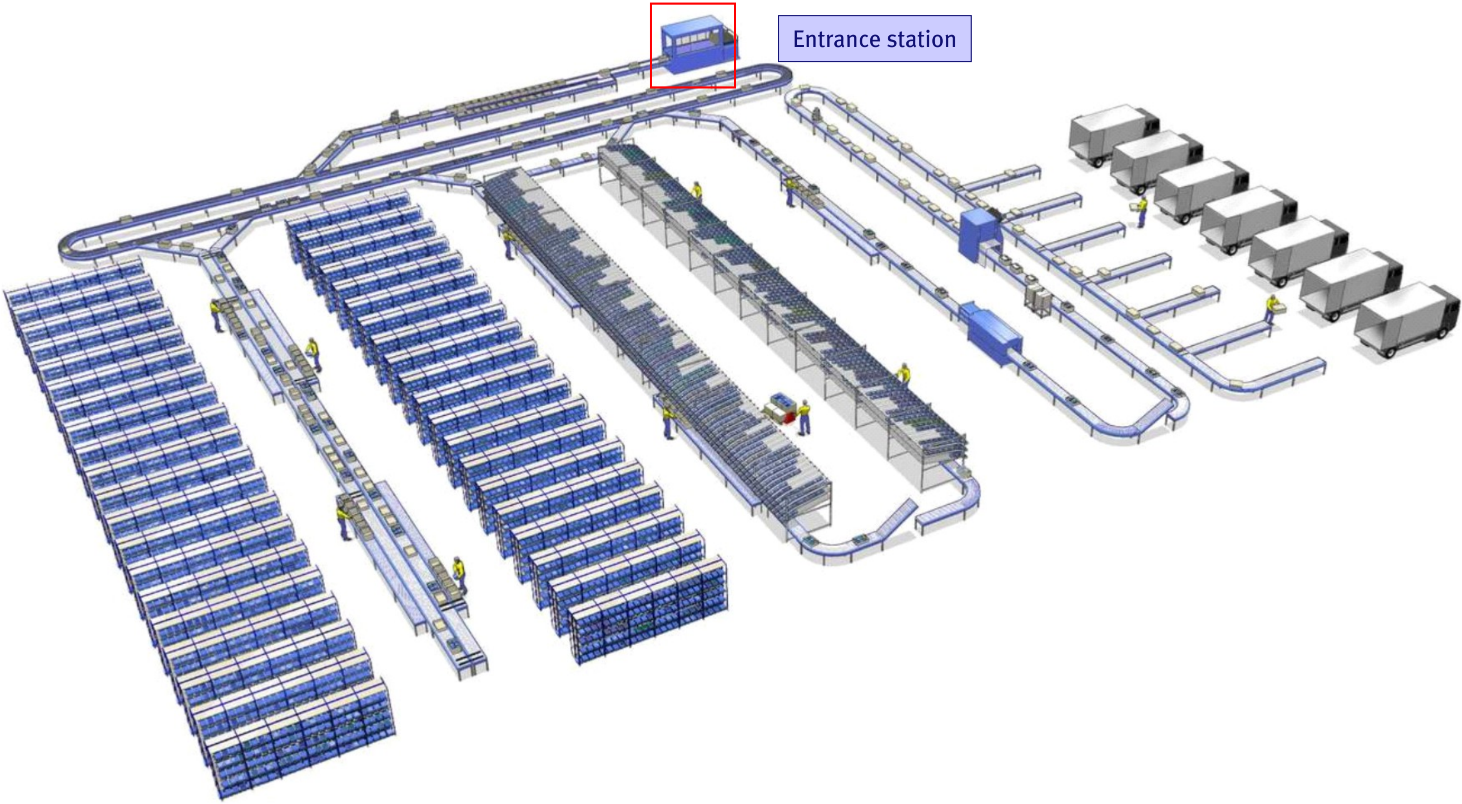

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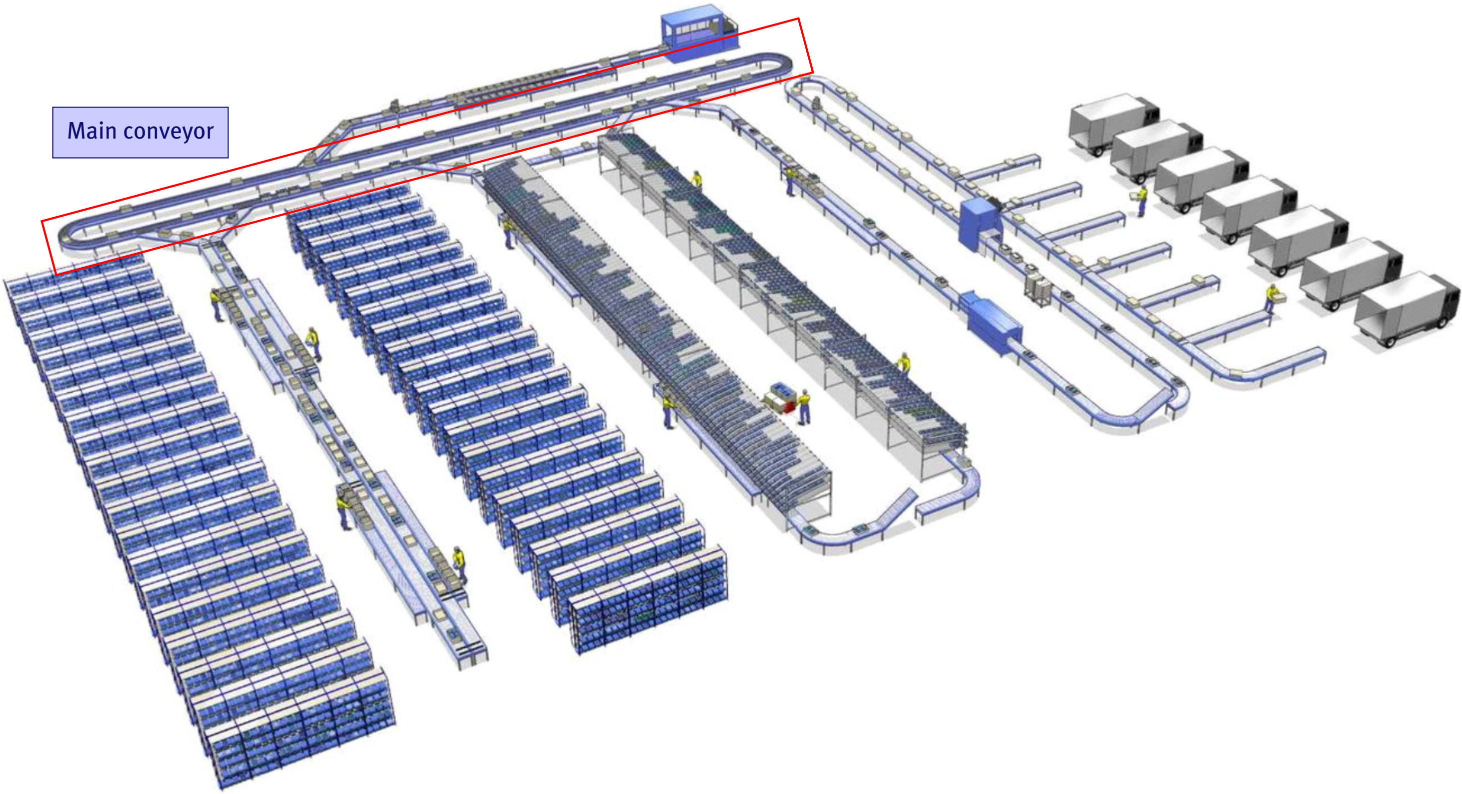
Zone-Picking Systems



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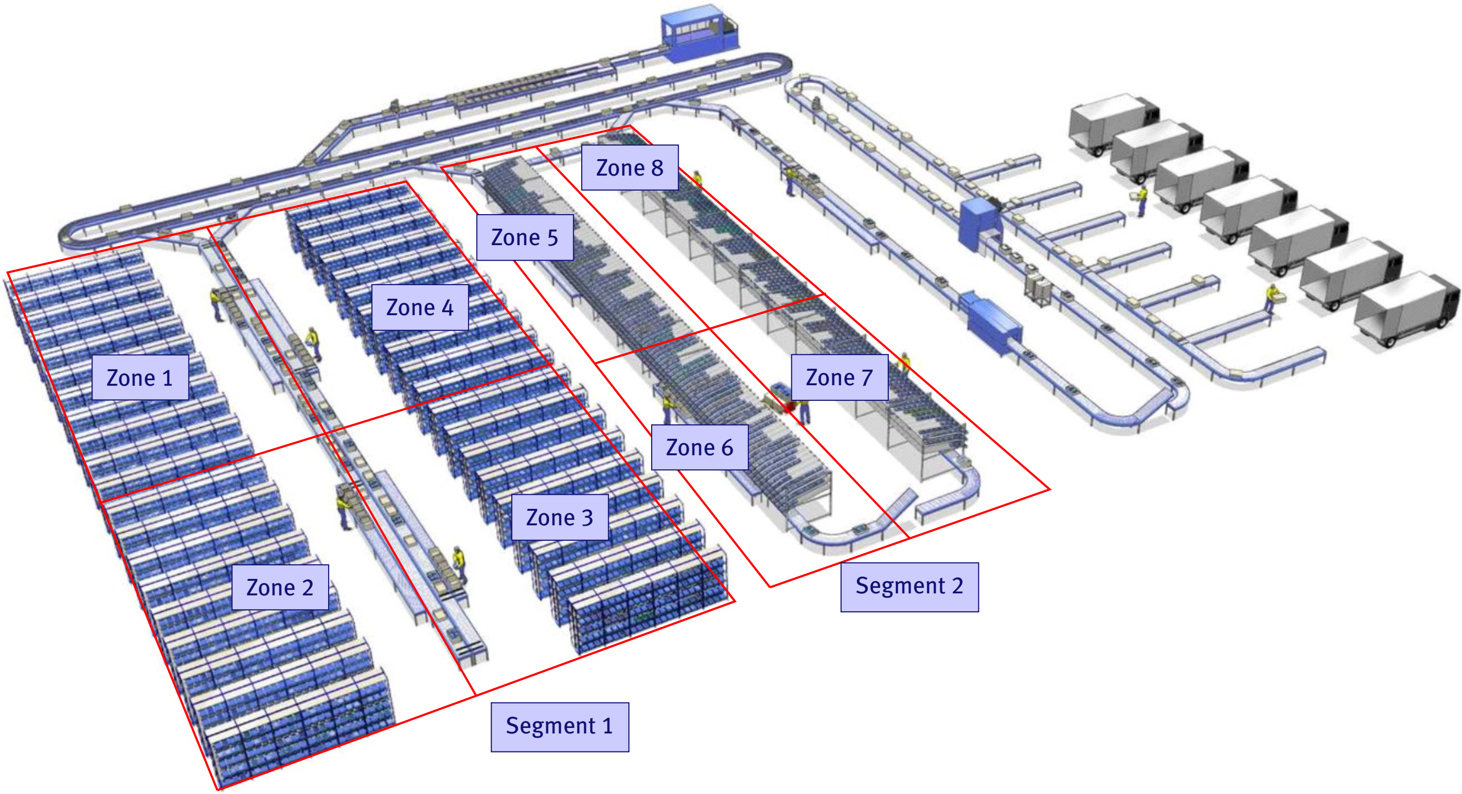

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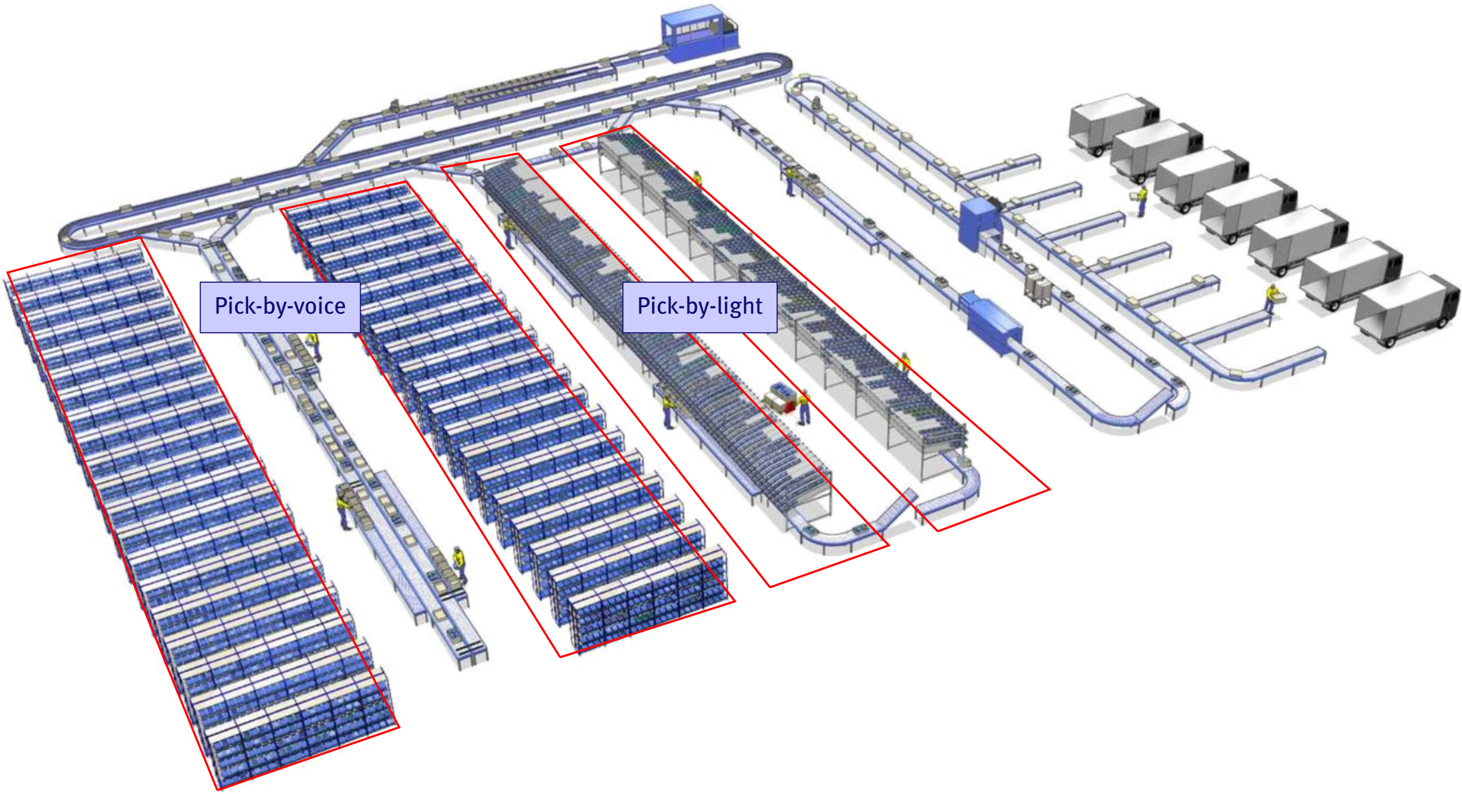

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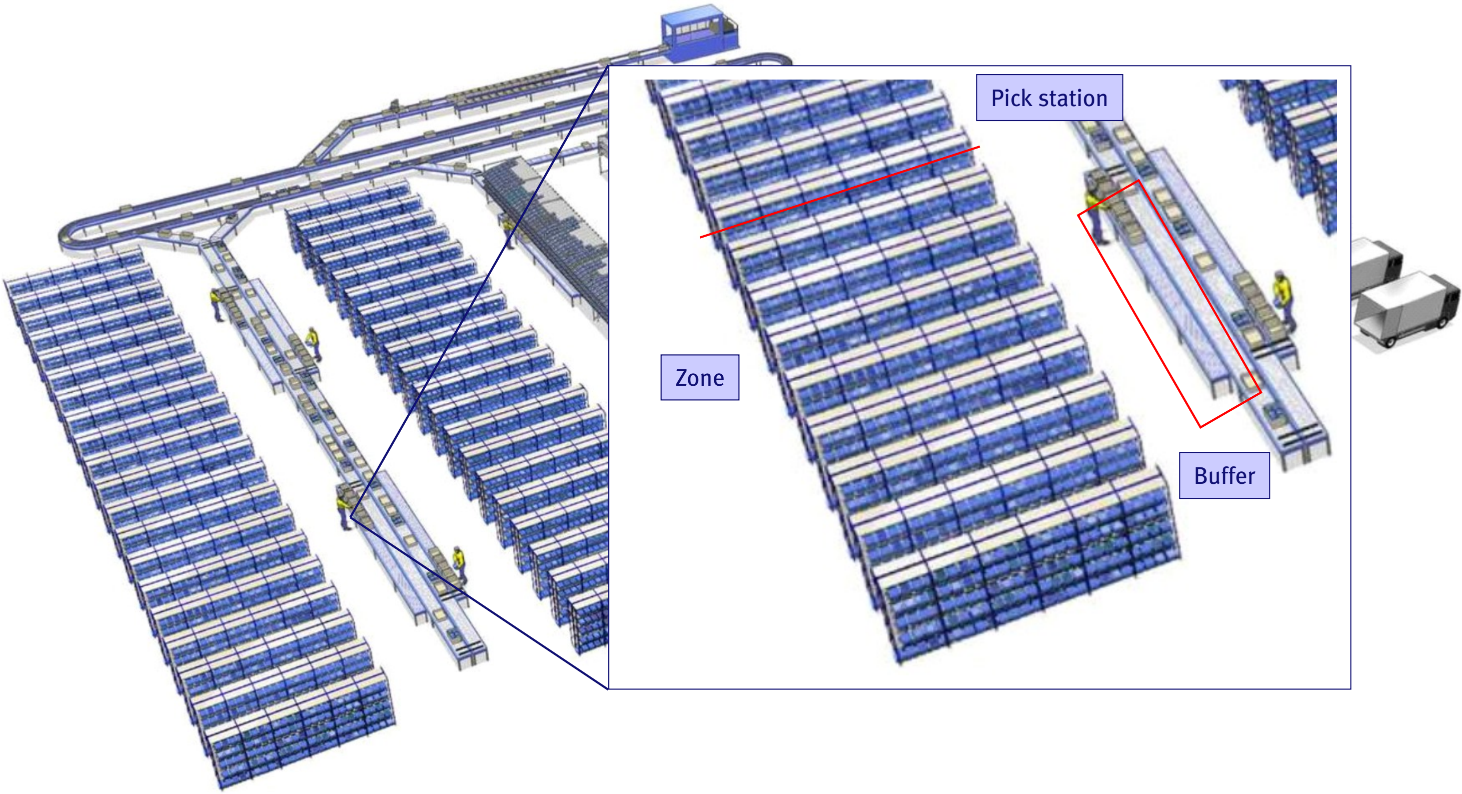

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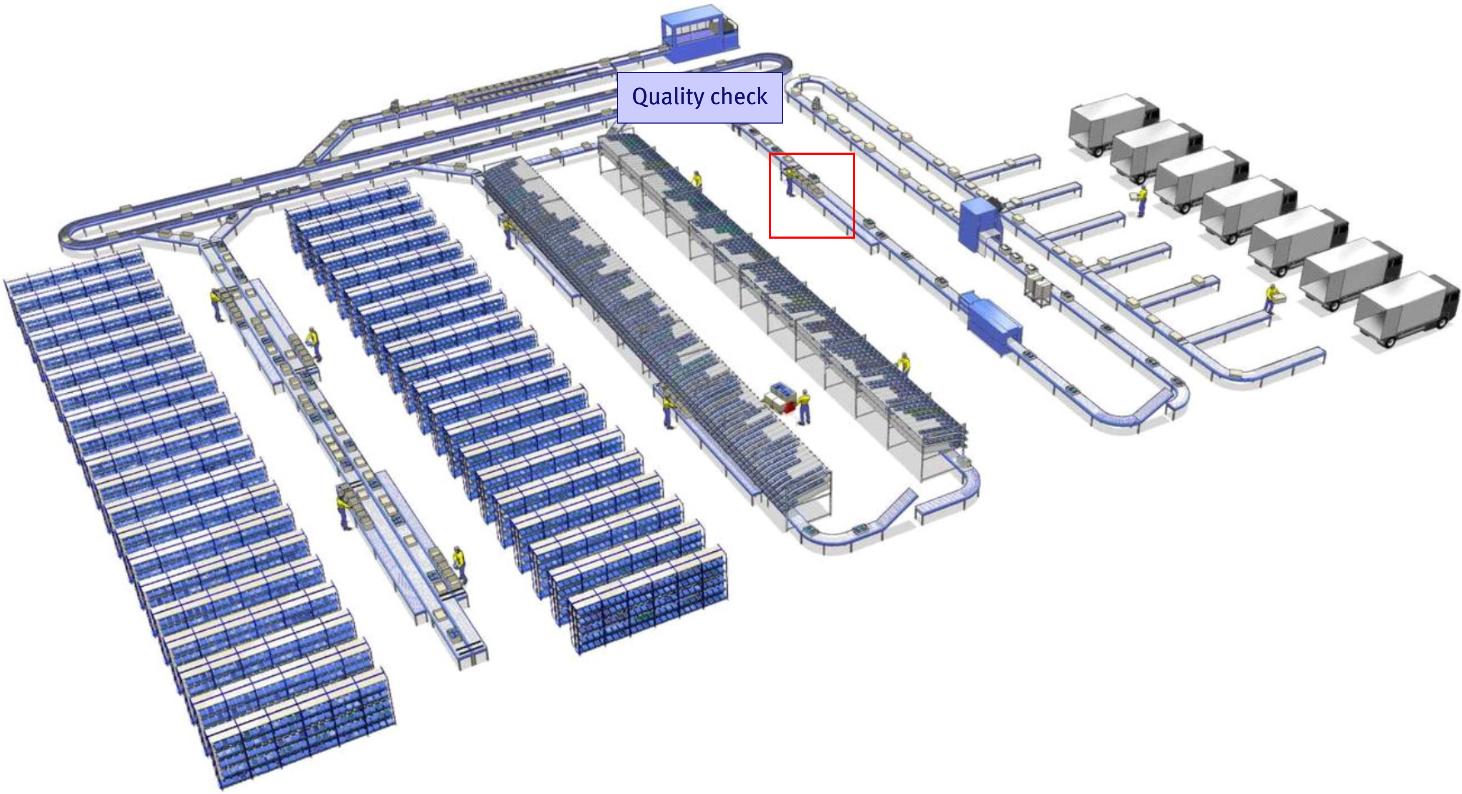

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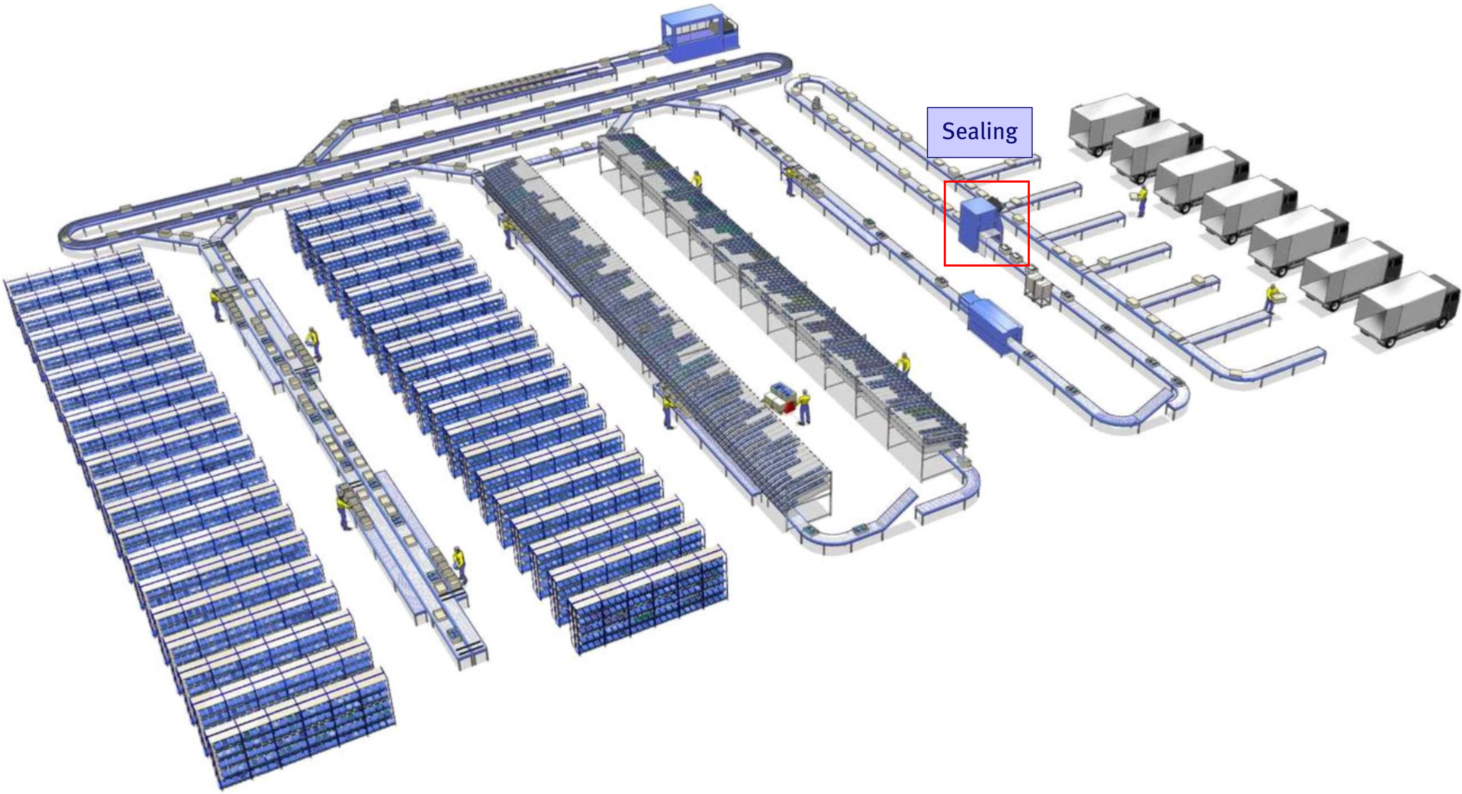

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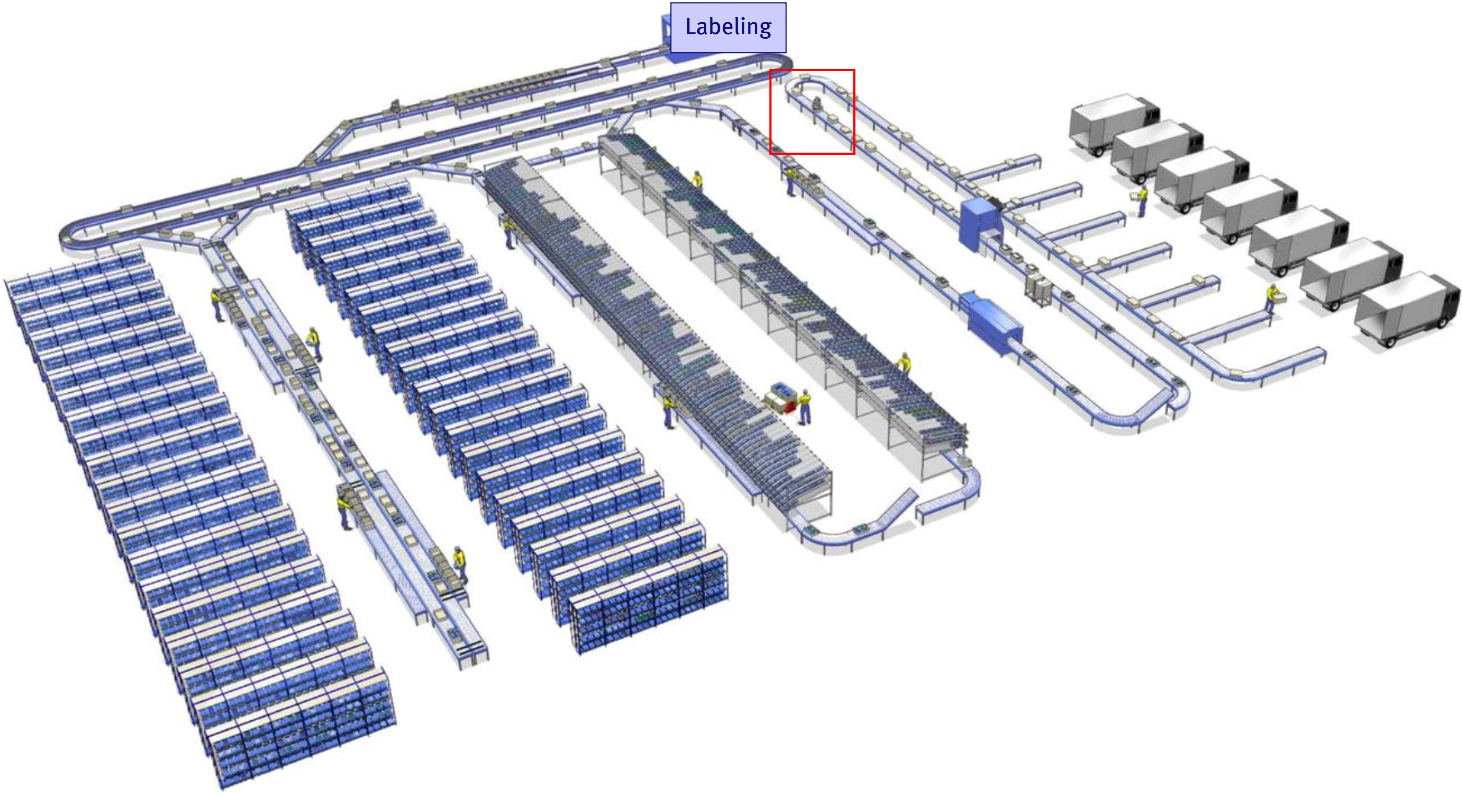

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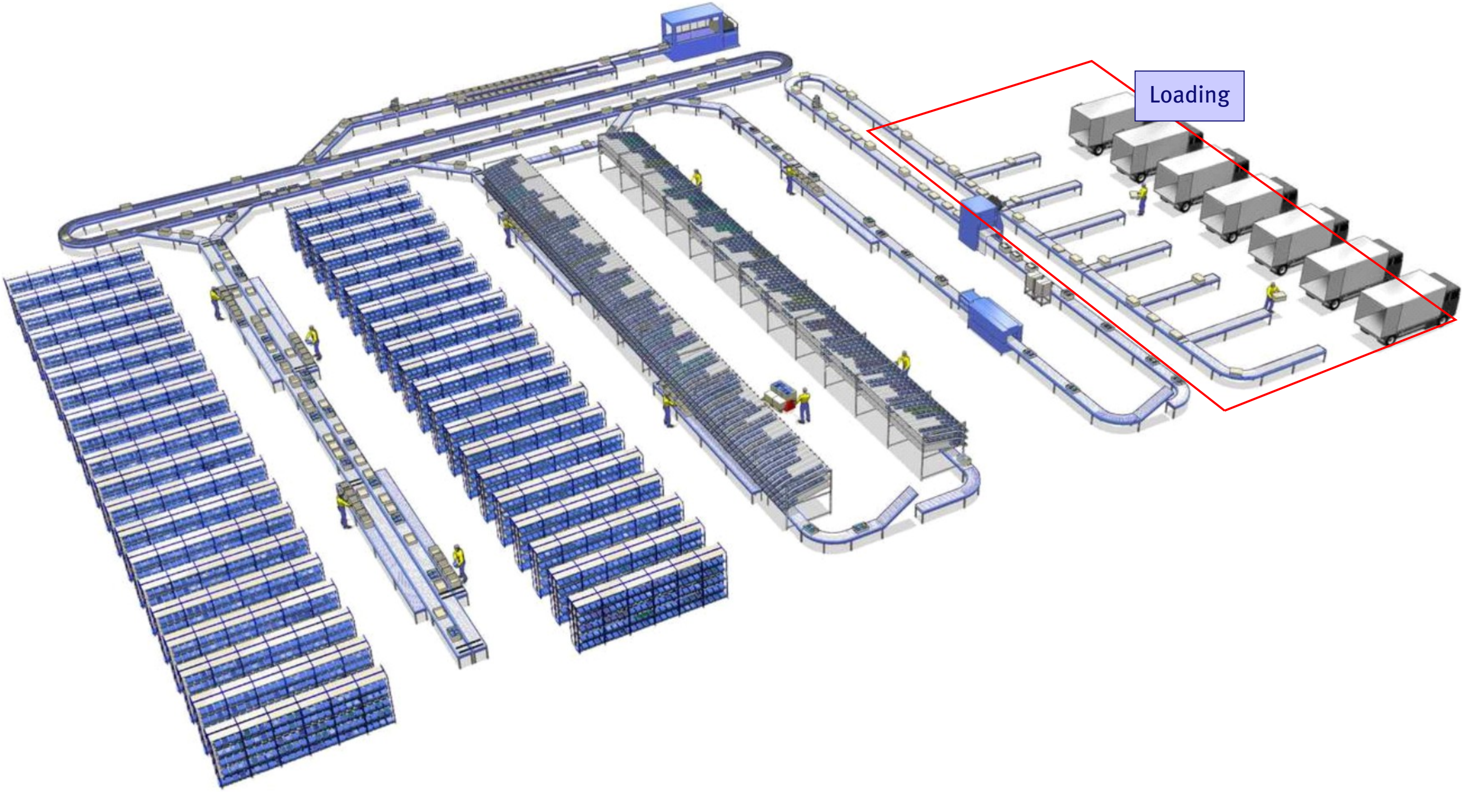
Zone-Picking Systems



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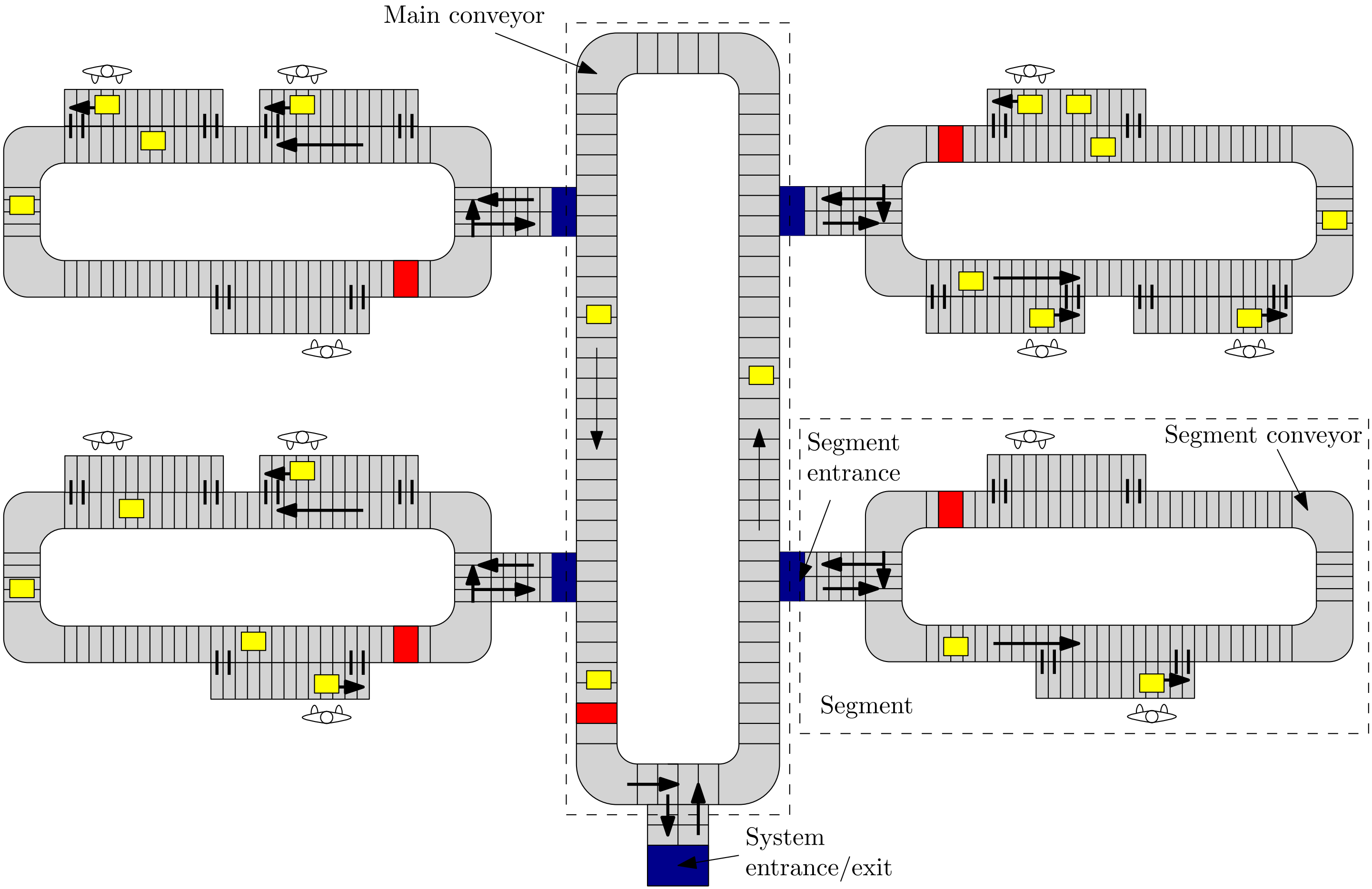

Zone-Picking Systems

How to design such complex systems?

- What should be the layout of the network?
- Size of zones?
- Where to locate items?
- What number of pickers and zones?
- Required WIP level?

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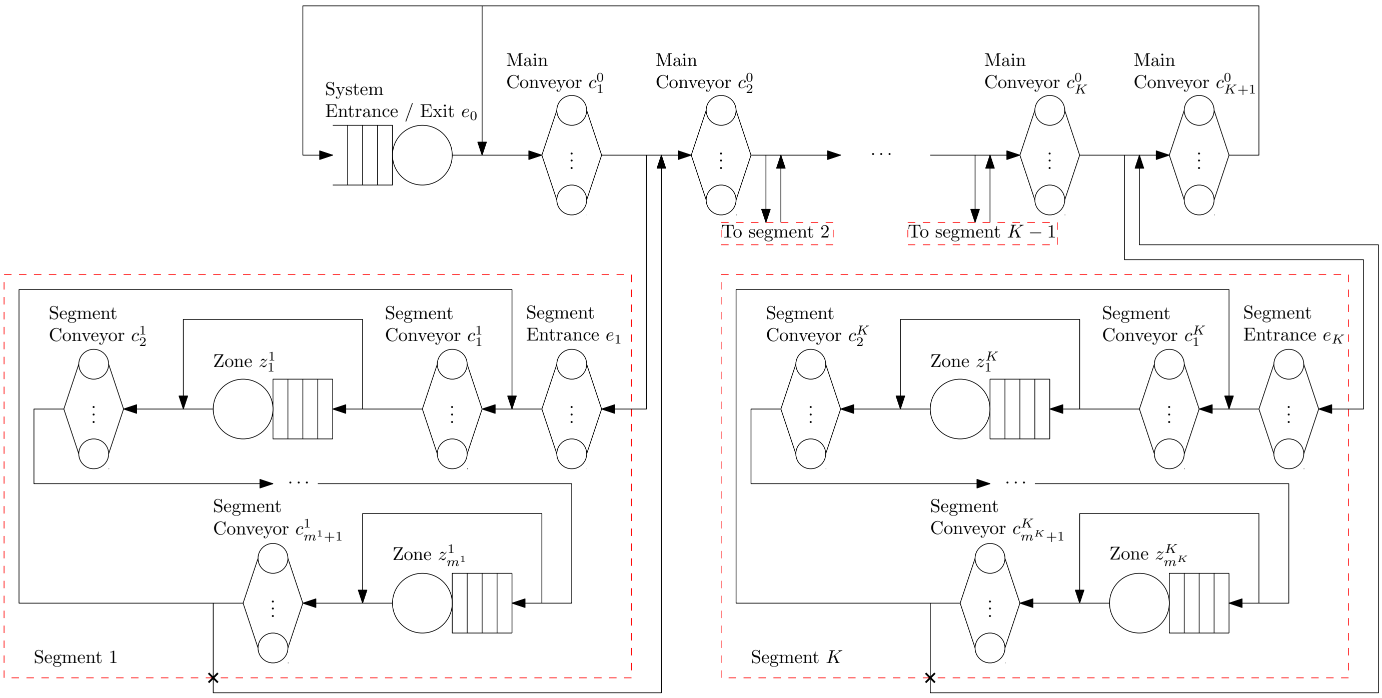
Zone-Picking Systems



Network model: implemented in Java Applet

Contents Basic Probability

- Sheldon M. Ross: *Probability Models*, Academic Press, 2003.
  - Chapter 1: 1.1-1.5
  - Chapter 2: 2.1-2.5
  - Chapter 3: 3.1-3.5
- Henk Tijms: *Understanding Probability*, Cambridge Univ. Press, 2012.
  - Chapter 7: 7.1, 7.2 (till 7.2.1), 7.3
  - Chapter 8: 8.1, 8.2
  - Chapter 9
  - Chapter 10: 10.1, 10.2, 10.3, 10.4 (till 10.4.8), 10.5, 10.6
  - Chapter 11: 11.1, 11.2, 11.3, 11.4.1, 11.5
  - Chapter 13: 13.1, 13.2, 13.3 (till 13.3.1)
  - Appendix:
    Permutations, Combinations, Exponential function, Geometric series

Basic Probability

Ingredients of a probability model:

- Sample space S:
  flipping a coin, rolling a die, process time, ...
- Events are (essentially) all subsets of S:
  $E = \{H\}$, $E = \{1, 2\}$, $E = (0, 1)$, ...
- Get new events by union, intersection, complement

Probabilities on Events

For each event E there is a number $P(E)$ such that:

- $0 \leq P(E) \leq 1$
- $P(S) = 1$
- $E_1, E_2, \ldots$ mutually exclusive, then $P(\bigcup_0^\infty E_i) = \sum_0^\infty P(E_i)$

Example: Tossing a coin, $P(\{H\}) = P(\{T\}) = \frac{1}{2}$
Example: Rolling a die, $P(\{1\}) = \frac{1}{6}$, $P(\{1, 2\}) = P(\{1\}) + P(\{2\}) = \frac{1}{3}$

Intuition:

If an experiment is repeated over and over, then, with probability 1, the long run portion of time that event E occurs is $P(E)$

Conditional probabilities

Probability of event E given that event F occurs,

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Example: Rolling a die twice, $P(\{i, j\}) = \frac{1}{36}$

Given that $i = 4$ (event F), what is probability that $j = 2$ (event E)?

$$P(E|F) = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$$

Example: Darts on unit disk $\{(x, y) | x^2 + y^2 \leq 1\}$

$$P(\text{distance to } 0 > \tfrac{1}{2} | x > 0) = \frac{\frac{1}{2}\pi - \frac{1}{8}\pi}{\frac{1}{2}\pi} = \frac{3}{4}$$

Note: $P(E \cap F) = P(E|F)P(F)$ and we usually write $P(E \cap F) = P(EF)$

Independent events

Events E and F are independent if

$$P(EF) = P(E)P(F)$$

and events $E_1, \ldots, E_n$ are independent is

$$P(E_1E_2\ldots E_n) = P(E_1)P(E_2)\cdots P(E_n)$$

Example: Rolling a die twice

$E =$ "$i + j = 6$" and $F =$ "$i = 4$". Independent?

$$P(E) = \frac{5}{36}, \quad P(F) = \frac{1}{6}, \quad P(EF) = \frac{1}{36}$$

Now $E =$ "$i + j = 7$". Independent?

Independent experiments: $S = S_1 \times S_2 \times \cdots \times S_n$ where

$$P(E_1E_2\ldots E_n) = P(E_1)P(E_2)\cdots P(E_n)$$

Random variable

Function of outcome: X

Example: Rolling a die twice

X is sum of outcomes, so $X = i + j$, $P(X = 2) = \frac{1}{36}$

Example: Flipping a coin indefinitely, $P(H) = p$

N is number of flips until first H (independent flips)

$$P(N = n) = (1 - p)^{n-1} p$$

Discrete random variable (rv): possible values are discrete

Continuous random variable: possible values are continuous

Distribution function

$$F(x) = P(X \le x), \quad x \in \mathbb{R}$$

Properties: $F(x) \uparrow$, $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$

$$P(y < X \le x) = F(x) - F(y)$$

Discrete random variable $X \in \{x_1, x_2, \ldots\}$

$$P(X = x_i) = p(x_i) > 0, \quad \sum_{i=1}^{\infty} p(x_i) = 1$$

Example: Bernoulli rv $P(X = 0) = 1 - P(X = 1) = 1 - p$ (1 is success)

Example: Binomial rv $X =$ #successes in n trials, p is success probability

$$p_i = P(X = i) = \binom{n}{i} p^i (1-p)^{n-i}, \quad i = 0, 1, \ldots n$$

Distribution function

Example: Geometric rv X = #trials till first success

$$p_n = P(X = n) = (1 - p)^{n-1} p, \quad n = 1, 2, 3, \ldots$$

Example: Poisson rv X

$$p_n = e^{-\lambda} \frac{\lambda^n}{n!}, \quad n = 0, 1, 2, \ldots$$

Continuous rv

X has a density $f(x)$

$$P(X \in B) = \int_B f(x)dx, \quad P(X \le b) = \int_{-\infty}^{b} f(x)dx,$$

so

$$\int_{-\infty}^{\infty} f(x)dx = 1, \quad P(a \le X \le b) = \int_a^b f(x)dx$$

Interpretation: $f(x)dx \approx P(x < X \le x + dx)$

$$F(x) = \int_{-\infty}^{x} f(y)dy, \quad \frac{d}{dx}F(x) = f(x), \quad P(X = b) = 0$$

Example: Uniform rv X on $[0, 1]$, or uniform rv X on $[a, b]$,

$$f(x) = 1, \quad 0 \le x \le 1$$

$$f(x) = 1/(b - a), \quad a \le x \le b$$

Continuous rv

Example: Normal rv X

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

Example: Exponential rv X with rate λ

$$f(x) = \lambda e^{-\lambda x}, \quad F(x) = 1 - e^{-\lambda x}, \quad x \geq 0$$

Memoryless property: X = lifetime of component

$$P(X > t + x | X > t) = \frac{P(X > t + x)}{P(X > t)} = e^{-\lambda x} = P(X > x)$$

so used is as good as new!

Failure rate $h(x)$

$$h(x)dx = P(x < X < x + dx | X > x) = \lambda dx$$

so constant failure rate

Expectation of rv X

Expected value of discrete rv X

$$E(X) = \sum_i P(X = x_i)x_i$$

Expected value of continuous rv X

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx \left(= \int_{-\infty}^{\infty} xdF(x) \right)$$

Examples:

Bernoulli	$E(X) = p$
Binomial	$E(X) = np$
Geometric	$E(X) = \frac{1}{p}$
Uniform$[0, 1]$	$E(X) = \frac{1}{2}$
Exponential	$E(X) = \frac{1}{\lambda}$
Normal	$E(X) = \mu$

Expectation of $g(X)$

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Example: X is exponential, $g(X) = X^2$

$$E(g(X)) = E(X^2) = \int_{-\infty}^{\infty} x^2 \lambda e^{-\lambda x} dx = \frac{2}{\lambda^2}$$

Property: Independent trials $X_1, X_2, \ldots$, then with probability 1

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \to E(X)$$

This is the (intuitively appealing) Strong Law of Large Numbers

Property:

$$E(aX + b) = aE(X) + b$$

Joint distributions of rv's

For rv's X, Y the joint distribution is

$$F(a, b) = P(X \leq a, Y \leq b)$$

and for continuous distribution with density $f(x, y)$

$$F(a, b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x, y) dx dy$$

Marginal distribution of X

$$F_X(a) = P(X \leq a) = \int_{-\infty}^{a} \int_{-\infty}^{\infty} f(x, y) dy dx = \int_{-\infty}^{a} f_X(x) dx$$

where

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

Joint distributions of rv's

Property: Linearity

$$E(aX + bY) = aE(X) + bE(Y)$$

Example: X_i Bernoulli, $X = X_1 + \cdots + X_n$ (Binomial)

$$E(X) = E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n) = np$$

Independent rv's

X and Y are independent if

$$P(X \leq a, Y \leq b) = P(X \leq a)P(Y \leq b)$$

or

$$F(a,b) = F_X(a)F_Y(b)$$

or

$$f(a,b) = f_X(a)f_Y(b)$$

Property: If X and Y are independent, then

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)f_Y(y)dxdy = E(X)E(Y)$$

Variance

Variance of X is (a measure of variability)

$$\text{var}(X) = E[(X - E(X))^2] = E(X^2) - (E(X))^2$$

Property: X and Y are independent, then

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y), \quad \text{var}(aX) = a^2\text{var}(X)$$

Definition: $X_1, X_2, \ldots, X_n$ are independent, then $\overline{X} = \frac{\sum_1^n X_i}{n}$ is sample mean

$$E(\overline{X}) = E(X), \quad \text{var}(\overline{X}) = \frac{\text{var}(X)}{n}$$

Example:

Bernouilli	$\text{var}(X) = p(1 - p)$
Binomial	$\text{var}(X) = np(1 - p)$
Exponential	$\text{var}(X) = \frac{1}{\lambda^2}$

Conditional expectation

Conditional probability

$$P(E|F) = \frac{P(EF)}{P(F)}$$

X, Y discrete rv's with
$p(x, y) = P(X = x, Y = y)$, $p_X(x) = P(X = x)$, $P_Y(y) = P(Y = y)$

Then conditional probability distribution of X given $Y = y$

$$p_{X|Y}(x|y) = P(X = x|Y = y) = \frac{p(x, y)}{p_Y(y)}$$

Conditional expectation

$$E(X|Y = y) = \sum_x x p_{X|Y}(x|y)$$

Conditional expectation

If X and Y are independent, then

$$p_{X|Y}(x|y) = P(X = x) = p_X(x), \quad E(X|Y = y) = E(X)$$

Example: X_1, X_2 binomial with n_1, p and n_2, p, and independent

$$P(X_1 = k|X_1 + X_2 = m) = \cdots = \frac{\binom{n_1}{k}\binom{n_2}{m-k}}{\binom{n_1+n_2}{m}}$$

which is the *Hypergeometric distribution*

Conditional expectation

X, Y continuous rv's with $f(x, y)$, $f_X(x)$, $f_Y(y)$

Then conditional density of X given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

Note

$$f_{X|Y}(x|y)dx = P(x < X \leq X + dx | y < Y \leq y + dy) = \frac{f(x, y)dxdy}{f_Y(y)dy}$$

Conditional expectation

$$E(X|Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx$$

Conditional expectation

Example:
$f(x, y) = \frac{1}{2}ye^{-xy}$ for $0 < x < \infty$, $0 < y < 2$, and $f(x, y) = 0$ otherwise

What is $E(e^{\frac{X}{2}}|Y = 1)$?

$$f_Y(y) = \int_0^\infty \frac{1}{2}ye^{-xy}dx = \frac{1}{2}$$

$$f_{X|Y}(x|y) = \frac{\frac{1}{2}ye^{-xy}}{\frac{1}{2}} = ye^{-yx}$$

so

$$E(e^{\frac{X}{2}}|Y = 1) = \int_0^\infty e^{\frac{x}{2}}e^{-x}dx = 2$$

Conditional expectation

Note: If X, Y are independent, then $E(X|Y=y) = E(X)$

Note: If $E(X|Y=y)$ is known, then

$$E(X) = \int_{-\infty}^{\infty} E(X|Y=y) f_Y(y) dy$$

or if Y is discrete

$$E(X) = \sum_y E(X|Y=y) p(Y=y)$$

Conditional expectation

Example: Total injuries per year

$$Y = \sum_{1}^{N} X_i$$

where N = #accidents per year, $E(N) = 4$, X_i = #workers injured, $E(X_i) = 2$ and X_i, N are all independent

Then

$$\begin{aligned} E(Y) &= E\left(\sum_{1}^{N} X_i\right) = \sum_{n=0}^{\infty} E\left(\sum_{1}^{N} X_i | N = n\right) P(N = n) \\ &= \sum_{n=0}^{\infty} E\left(\sum_{1}^{n} X_i | N = n\right) P(N = n) = \sum_{n=0}^{\infty} E\left(\sum_{1}^{n} X_i\right) P(N = n) \\ &= \sum_{n=0}^{\infty} nE(X)P(N = n) = E(X)E(N) = 2E(N) = 8 \end{aligned}$$

Conditional expectation

Example: Getting a seat in a train

Y is distance to closest door, Y is Uniform$[0, 2]$

If distance is $Y = y$, the probability of getting a seat is $1 - \sqrt{\frac{1}{2}y}$

What is probability of getting a seat?
(which is NOT equal to $1 - \sqrt{\frac{1}{2} \cdot E(Y)} = 1 - \sqrt{\frac{1}{2} \cdot 1} = 0.293$)

Let $X = 1$ if success (get a seat), and $X = 0$ otherwise

$$P(X = 1) = \int_0^2 P(X = 1|Y = y)\frac{1}{2}dy = \int_0^2 (1 - \sqrt{\frac{1}{2}y})\frac{1}{2}dy = \frac{1}{3}$$