Stochastic Models of Manufacturing Systems

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Organization

- 7 lectures (lecture of May 12 is canceled)
- Studyguide available (with notes, slides, assignments, references), see http://www.win.tue.nl/~iadan/4t400
- Examination consists of:
  - Weekly (8) take home assignments
  - Send in take home assignments individually
  - Best 7 out of 8 take home assignments count (40%)
  - Take home assignments of last week will be discussed in BZ
  - Final assignment, done in groups of two (60%)
• Basic probability (refresher)
• Basic statistics for discrete-event simulation
• Modeling and analysis of manufacturing systems:
  – Single-stage systems
  – Multi-stage flow lines
  – Job-shop systems
  – CONWIP systems
Some basic steps:

- Identify the issues to be addressed
- Learn about the system
- Choose a modeling approach
- Develop and test the model
- Verify and validate the model
- Experiment with the model
- Present the results
Various types of models:
- Physical models
- Simulation models
- Analytical models

But why modeling?
- Understanding
- Improvement
- Optimization
- Decision making
Some issues:

- Complexity versus Simplicity
- Flexibility
- Data requirements
- Transparency

Analytical and simulation capability: Effective modeling requires both!
Servers are equally fast, 10 circulating jobs

**Question:** Replace one server by a server that is twice as fast. How does this affect average throughput time? Throughtput?

**Question:** Does your answer change in case of more jobs? Less jobs?
Multiple Machines or a Single One?

4 machines, or one machine that is four times faster?

**Question:** What do you prefer, 4 machines or one fast machine?

**Question:** What do you prefer if process time variability is high?

**Question:** What do you prefer if the load is low?
Multiple Machines or a Single One?

4 machines, or one machine that is four times faster?

The required information is captured in the following formula:

\[ E(S) \approx \frac{\prod W}{1 - \rho} \frac{E(R)}{c} + E(B) \]
Running example

A robotic dairy barn
How to design such a barn?
Zone-Picking Systems
Zone-Picking Systems
Zone-Picking Systems
Zone-Picking Systems
Zone-Picking Systems
Zone-Picking Systems
Zone-Picking Systems

Main conveyor

Segment entrance

Segment conveyor

Segment

System entrance/exit

Zone-Picking Systems
Zone-Picking Systems

How to design such complex systems?

- What should be the layout of the network?
- Size of zones?
- Where to locate items?
- What number of pickers and zones?
- Required WIP level?
Zone-Picking Systems

Network model: implemented in Java Applet
- Chapter 1: 1.1-1.5
- Chapter 2: 2.1-2.5
- Chapter 3: 3.1-3.5

- Chapter 7: 7.1, 7.2 (till 7.2.1), 7.3
- Chapter 8: 8.1, 8.2
- Chapter 9
- Chapter 10: 10.1, 10.2, 10.3, 10.4 (till 10.4.8), 10.5, 10.6
- Chapter 11: 11.1, 11.2, 11.3, 11.4.1, 11.5
- Chapter 13: 13.1, 13.2, 13.3 (till 13.3.1)
- Appendix:
  Permutations, Combinations, Exponential function, Geometric series
Ingredients of a probability model:

- Sample space $S$: flipping a coin, rolling a die, process time, ...
- Events are (essentially) all subsets of $S$: $E = \{H\}$, $E = \{1, 2\}$, $E = (0, 1)$, ...
- Get new events by union, intersection, complement
For each event $E$ there is a number $P(E)$ such that:

- $0 \leq P(E) \leq 1$
- $P(S) = 1$
- $E_1, E_2, \ldots$ mutually exclusive, then $P(\bigcup_{i=0}^{\infty} E_i) = \sum_{i=0}^{\infty} P(E_i)$

**Example:** Tossing a coin, $P(\{H\}) = P(\{T\}) = \frac{1}{2}$
**Example:** Rolling a die, $P(\{1\}) = \frac{1}{6}$, $P(\{1, 2\}) = P(\{1\}) + P(\{2\}) = \frac{1}{3}$

**Intuition:**

If an experiment is repeated over and over, then, with probability 1, the long run portion of time that event $E$ occurs is $P(E)$
Conditional probabilities

Probability of event $E$ given that event $F$ occurs,$$
P(E|F) = \frac{P(E \cap F)}{P(F)}$$

**Example:** Rolling a die twice, $P(\{i, j\}) = \frac{1}{36}$

Given that $i = 4$ (event $F$), what is probability that $j = 2$ (event $E$)?

$$P(E|F) = \frac{1}{36} \cdot \frac{1}{6} = \frac{1}{6}$$

**Example:** Darts on unit disk $\{(x, y)|x^2 + y^2 \leq 1\}$

$$P(\text{distance to } 0 > \frac{1}{2}|x > 0) = \frac{\frac{1}{2}\pi - \frac{1}{8}\pi}{\frac{1}{2}\pi} = \frac{3}{4}$$

**Note:** $P(E \cap F) = P(E|F)P(F)$ and we usually write $P(E \cap F) = P(EF)$
Events $E$ and $F$ are independent if

$$P(EF) = P(E)P(F)$$

and events $E_1, \ldots, E_n$ are independent is

$$P(E_1E_2\ldots E_n) = P(E_1)P(E_2)\cdots P(E_n)$$

Example: Rolling a die twice

$E =$ "$i + j = 6$" and $F =$ "$i = 4$". Independent?

$$P(E) = \frac{5}{36}, \quad P(F) = \frac{1}{6}, \quad P(EF) = \frac{1}{36}$$

Now $E =$ "$i + j = 7$". Independent?

Independent experiments: $S = S_1 \times S_2 \times \cdots \times S_n$ where

$$P(E_1E_2\ldots E_n) = P(E_1)P(E_2)\cdots P(E_n)$$
Function of outcome: $X$

**Example:** Rolling a die twice

$X$ is sum of outcomes, so $X = i + j$, $P(X = 2) = \frac{1}{36}$

**Example:** Flipping a coin indefinitely, $P(H) = p$

$N$ is number of flips until first $H$ (independent flips)

$$P(N = n) = (1 - p)^{n-1}p$$

**Discrete** random variable (rv): possible values are discrete

**Continuous** random variable: possible values are continuous
\[ F(x) = P(X \leq x), \quad x \in \mathbb{R} \]

**Properties:** \( F(x) \uparrow, \lim_{x \to \infty} F(x) = 1, \lim_{x \to -\infty} F(x) = 0 \)

\[ P(y < X \leq x) = F(x) - F(y) \]

**Discrete random variable** \( X \in \{x_1, x_2, \ldots\} \)

\[ P(X = x_i) = p(x_i) > 0, \quad \sum_{i=1}^{\infty} p(x_i) = 1 \]

**Example:** Bernoulli rv \( P(X = 0) = 1 - P(X = 1) = 1 - p \) (1 is success)

**Example:** Binomial rv \( X = \#\text{successes in } n \text{ trials}, \ p \text{ is success probability} \)

\[ p_i = P(X = i) = \binom{n}{i} p^i (1 - p)^{n-i}, \quad i = 0, 1, \ldots n \]
Example: Geometric rv $X = \#\text{trials till first success}$

$$p_n = P(X = n) = (1 - p)^{n-1} p, \quad n = 1, 2, 3, \ldots$$

Example: Poisson rv $X$

$$p_n = e^{-\lambda}\frac{\lambda^n}{n!}, \quad n = 0, 1, 2, \ldots$$
Continuous rv

$X$ has a density $f(x)$

$$P(X \in B) = \int_B f(x)dx, \quad P(X \leq b) = \int_{-\infty}^b f(x)dx,$$

so

$$\int_{-\infty}^\infty f(x)dx = 1, \quad P(a \leq X \leq b) = \int_a^b f(x)dx$$

**Interpretation:** $f(x)dx \approx P(x < X \leq x + dx)$

$$F(x) = \int_{-\infty}^x f(y)dy, \quad \frac{d}{dx}F(x) = f(x), \quad P(X = b) = 0$$

**Example:** Uniform rv $X$ on $[0, 1]$, or uniform rv $X$ on $[a, b]$,

$$f(x) = 1, \quad 0 \leq x \leq 1$$

$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$
Example: Normal rv $X$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

Example: Exponential rv $X$ with rate $\lambda$

$$f(x) = \lambda e^{-\lambda x}, \quad F(x) = 1 - e^{-\lambda x}, \quad x \geq 0$$

Memoryless property: $X$ = lifetime of component

$$P(X > t + x | X > t) = \frac{P(X > t + x)}{P(X > t)} = e^{-\lambda x} = P(X > x)$$

so used is as good as new!

Failure rate $h(x)$

$$h(x)dx = P(x < X < x + dx | X > x) = \lambda dx$$

so constant failure rate
Expected value of discrete rv $X$

$$E(X) = \sum_i P(X = x_i) x_i$$

Expected value of continuous rv $X$

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx \left( = \int_{-\infty}^{\infty} x dF(x) \right)$$

Examples:

- Bernoulli $E(X) = p$
- Binomial $E(X) = np$
- Geometric $E(X) = \frac{1}{p}$
- Uniform $[0, 1]$ $E(X) = \frac{1}{2}$
- Exponential $E(X) = \frac{1}{\lambda}$
- Normal $E(X) = \mu$
Expectation of $g(X)$

\[ E(g(X)) = \int_{-\infty}^{\infty} g(x)f_X(x)\,dx \]

**Example:** $X$ is exponential, $g(X) = X^2$

\[ E(g(X)) = E(X^2) = \int_{-\infty}^{\infty} x^2 e^{-\lambda x} \,dx = \frac{2}{\lambda^2} \]

**Property:** Independent trials $X_1, X_2, \ldots$, then with probability 1

\[ \frac{X_1 + X_2 + \cdots + X_n}{n} \to E(X) \]

This is the (intuitively appealing) Strong Law of Large Numbers

**Property:**

\[ E(aX + b) = aE(X) + b \]
Joint distributions of rv’s

For rv’s $X$, $Y$ the joint distribution is

$$F(a, b) = P(X \leq a, Y \leq b)$$

and for continuous distribution with density $f(x, y)$

$$F(a, b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x, y) \, dx \, dy$$

**Marginal distribution of $X$**

$$F_X(a) = P(X \leq a) = \int_{-\infty}^{a} \int_{-\infty}^{\infty} f(x, y) \, dy \, dx = \int_{-\infty}^{a} f_X(x) \, dx$$

where

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy$$
Property: Linearity

\[ E(aX + bY) = aE(X) + bE(Y) \]

Example: \( X_i \) Bernoulli, \( X = X_1 + \cdots + X_n \) (Binomial)

\[ E(X) = E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n) = np \]
$X$ and $Y$ are independent if

$$P(X \leq a, Y \leq b) = P(X \leq a)P(Y \leq b)$$

or

$$F(a, b) = F_X(a)F_Y(b)$$

or

$$f(a, b) = f_X(a)f_Y(b)$$

**Property:** If $X$ and $Y$ are independent, then

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf_X(x)f_Y(y)dxdy = E(X)E(Y)$$
Variance

Variance of $X$ is (a measure of variability)

$$\text{var}(X) = E[(X - E(X))^2] = E(X^2) - (E(X))^2$$

Property: $X$ and $Y$ are independent, then

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y), \quad \text{var}(aX) = a^2\text{var}(X)$$

Definition: $X_1, X_2, \ldots, X_n$ are independent, then $\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ is sample mean

$$E(\overline{X}) = E(X), \quad \text{var}(\overline{X}) = \frac{\text{var}(X)}{n}$$

Example:

- Bernouilli $\quad \text{var}(X) = p(1 - p)$
- Binomial $\quad \text{var}(X) = np(1 - p)$
- Exponential $\quad \text{var}(X) = \frac{1}{\lambda^2}$
Conditional probability

\[ P(E|F) = \frac{P(EF)}{P(F)} \]

\( X, Y \) discrete rv's with
\[ p(x, y) = P(X = x, Y = y), \quad p_X(x) = P(X = x), \quad P_Y(y) = P(Y = y) \]

Then conditional probability distribution of \( X \) given \( Y = y \)

\[ p_{X|Y}(x|y) = P(X = x|Y = y) = \frac{p(x, y)}{p_Y(y)} \]

Conditional expectation

\[ E(X|Y = y) = \sum_x x p_{X|Y}(x|y) \]
Conditional expectation

If $X$ and $Y$ are independent, then

$$p_{X|Y}(x|y) = P(X = x) = p_X(x), \quad E(X|Y = y) = E(X)$$

Example: $X_1$, $X_2$ binomial with $n_1$, $p$ and $n_2$, $p$, and independent

$$P(X_1 = k|X_1 + X_2 = m) = \cdots = \frac{{n_1 \choose k} {n_2 \choose m-k}}{{n_1 + n_2 \choose m}}$$

which is the Hypergeometric distribution
Conditional expectation

$X, Y$ continuous rv’s with $f(x, y), f_X(x), f_Y(y)$

Then conditional density of $X$ given $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}$$

Note

$$f_{X|Y}(x|y)dx = P(x < X \leq X + dx|y < Y \leq y + dy) = \frac{f(x, y)dxdy}{f_Y(y)dy}$$

Conditional expectation

$$E(X|Y = y) = \int_{-\infty}^{\infty} xf_{X|Y}(x|y)dx$$
Conditional expectation

Example:
\[ f(x, y) = \frac{1}{2}ye^{-xy} \text{ for } 0 < x < \infty, 0 < y < 2, \text{ and } f(x, y) = 0 \text{ otherwise} \]

What is \( E(e^{\frac{X}{2}}|Y = 1) \)?

\[
f_Y(y) = \int_0^\infty \frac{1}{2} ye^{-xy} \, dx = \frac{1}{2}
\]

\[
f_{X|Y}(x|y) = \frac{\frac{1}{2} ye^{-xy}}{\frac{1}{2}} = ye^{-yx}
\]

So

\[
E(e^{\frac{X}{2}}|Y = 1) = \int_0^\infty e^{\frac{x}{2}} e^{-x} \, dx = 2
\]
Note: If $X, Y$ are independent, then $E(X|Y = y) = E(X)$

Note: If $E(X|Y = y)$ is known, then

$$E(X) = \int_{-\infty}^{\infty} E(X|Y = y) f_Y(y) dy$$

or if $Y$ is discrete

$$E(X) = \sum_y E(X|Y = y) p(Y = y)$$
Example: Total injuries per year

\[ Y = \sum_{1}^{N} X_i \]

where \( N = \# \text{accidents per year}, \ E(N) = 4, \ X_i = \# \text{workers injured}, \ E(X_i) = 2 \)

and \( X_i, N \) are all independent

Then

\[
E(Y) = E \left( \sum_{1}^{N} X_i \right) = \sum_{n=0}^{\infty} E \left( \sum_{1}^{N} X_i | N = n \right) P(N = n)
\]

\[
= \sum_{n=0}^{\infty} E \left( \sum_{1}^{n} X_i | N = n \right) P(N = n) = \sum_{n=0}^{\infty} E \left( \sum_{1}^{n} X_i \right) P(N = n)
\]

\[
= \sum_{n=0}^{\infty} n E(X) P(N = n) = E(X) E(N) = 2E(N) = 8
\]
Example: Getting a seat in a train

$Y$ is distance to closest door, $Y$ is Uniform$[0, 2]$

If distance is $Y = y$, the probability of getting a seat is $1 - \sqrt{\frac{1}{2}y}$

What is probability of getting a seat?

(which is NOT equal to $1 - \sqrt{\frac{1}{2} \cdot E(Y)} = 1 - \sqrt{\frac{1}{2} \cdot 1} = 0.293$)

Let $X = 1$ if success (get a seat), and $X = 0$ otherwise

$$P(X = 1) = \int_{0}^{2} P(X = 1|Y = y) \frac{1}{2} dy = \int_{0}^{2} (1 - \sqrt{\frac{1}{2}y}) \frac{1}{2} dy = \frac{1}{3}$$