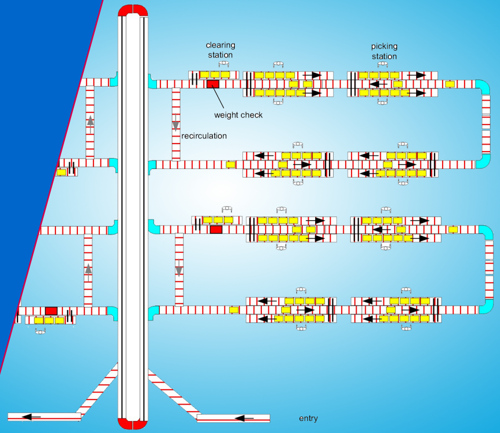
Stochastic Models of Manufacturing Systems

Ivo Adan



Technische Universiteit
Eindhoven
University of Technology

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Fundamental relations

Little's law

Consider system in equilibrium

- $E(L)$ is mean number in system
- $E(S)$ is mean time spent in system
- λ is arrival rate (or departure rate)

Then:

$$E(L) = \lambda E(S)$$

The definition of system is flexible (e.g. queue, server, queue+server)

PASTA: Poisson Arrivals See Time Averages

Poisson arrivals see the system in equilibrium:
They see the same as random outside observer!

Single server: exponential

- Poisson arrivals with rate λ
- Exponential service times with mean $1/\mu$
- Stability: $\lambda < \mu$ or $\rho = \lambda/\mu < 1$
- Single server and FCFS service

Then:

- $p_k = P(k \text{ in system}) = (1-\rho)\rho^k$, $k = 0, 1, \ldots$ (Geometric)
- $E(L) = \sum_{k=0}^{\infty} k p_k = (1-\rho)\rho \sum_{k=0}^{\infty} k \rho^{k-1} = (1-\rho)\rho \frac{1}{(1-\rho)^2} = \frac{\rho}{1-\rho}$
- $E(S) = E(L)/\lambda = \frac{1/\mu}{1-\rho}$
- $E(Q) = \sum_{k=1}^{\infty} (k-1) p_k = \frac{\rho^2}{1-\rho}$
- $E(W) = E(Q)/\lambda = \frac{\rho/\mu}{1-\rho}$

or via PASTA+Little...

Single server: exponential

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Via PASTA+Little...

$$E(S) = E(L^a)\frac{1}{\mu} + \frac{1}{\mu},$$

where L^a is the number on arrival. By PASTA, $E(L^a) = E(L)$, so

$$E(S) = E(L)\frac{1}{\mu} + \frac{1}{\mu}$$

and thus by Little's law, $E(L) = \lambda E(S)$,

$$E(S) = \frac{1/\mu}{1-\rho}.$$

Single server: exponential

- Poisson arrivals with rate λ
- Exponential service times with mean $1/\mu$
- Stability: $\lambda < \mu$ or $\rho = \lambda/\mu < 1$
- Single server and FCFS service

Then:

- Let a_k be arrival distribution, then by PASTA
  $a_k = P(k$ in system just before arrival$) = p_k$
- Let d_k be departure distribution, then
  $d_k = P(k$ in system just after departure$) = a_k$ $(= p_k)$
- $P(S > t) = e^{-\mu(1-\rho)t}$, $t \geq 0$ (Exponential)
- $P(W > t) = \rho e^{-\mu(1-\rho)t}$, $t \geq 0$ (nearly Exponential)
- Output process is again Poisson!

Single server: exponential

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Let $f_{k+1}(t)$ be density of $k+1$ independent exponential service times,

$$f_{k+1}(t) = \mu e^{-\mu t} \frac{(\mu t)^k}{k!} \quad \text{(Erlang-}k+1\text{ distribution)}.$$

Density $f_S(t)$ of sojourn (waiting plus service) time S:

$$\begin{aligned} f_S(t) &= \sum_{k=0}^{\infty} a_k f_{k+1}(t) = \sum_{k=0}^{\infty} (1-\rho)\rho^k \mu e^{-\mu t} \frac{(\mu t)^k}{k!} \\ &= \mu(1-\rho)e^{-\mu t} \sum_{k=0}^{\infty} \frac{(\rho\mu t)^k}{k!} = \mu(1-\rho)e^{-\mu(1-\rho)t}, \end{aligned}$$

so S is exponential with parameter $(1-\rho)\mu$!

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Density $f_W(t)$ of waiting time W:

$$f_W(t) = \sum_{k=1}^{\infty} a_k f_k(t) = \sum_{k=1}^{\infty} (1-\rho)\rho^k \mu e^{-\mu t} \frac{(\mu t)^{k-1}}{(k-1)!}$$

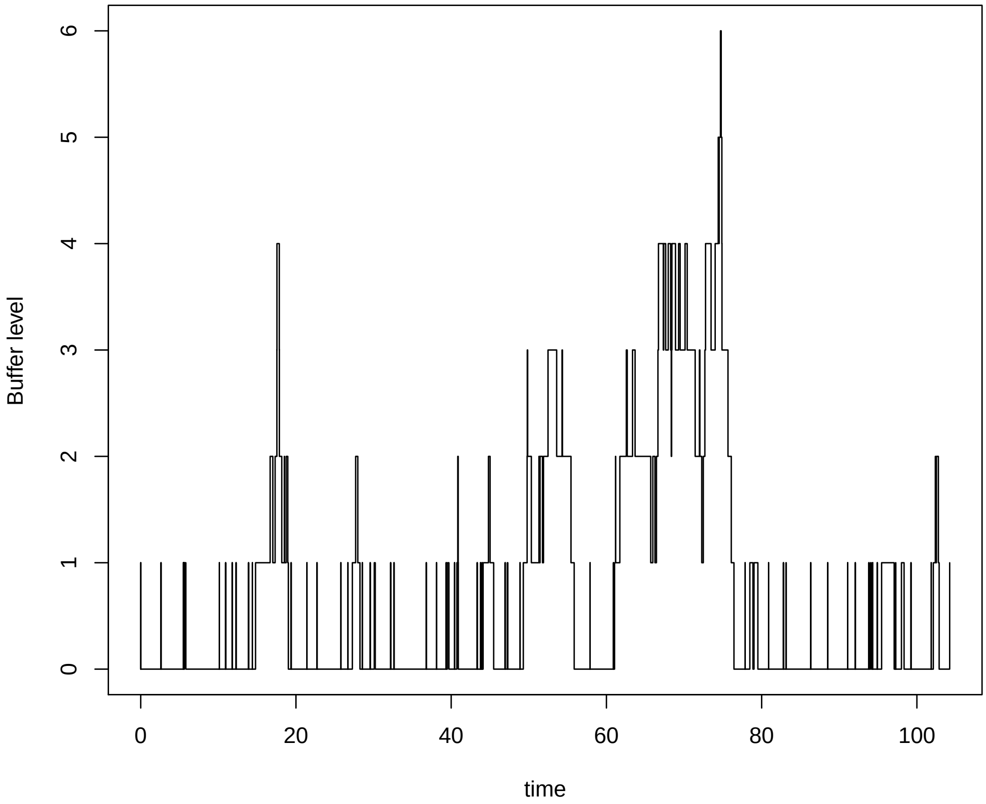
$$= \rho\mu(1-\rho)e^{-\mu t} \sum_{k=1}^{\infty} \frac{(\rho\mu t)^{k-1}}{(k-1)!} = \rho\mu(1-\rho)e^{-\mu(1-\rho)t}.$$

Note that $P(W > 0) = \rho$ by PASTA, so

$$P(W > t | W > 0) = \frac{P(W > t)}{P(W > 0)} = e^{-\mu(1-\rho)t},$$

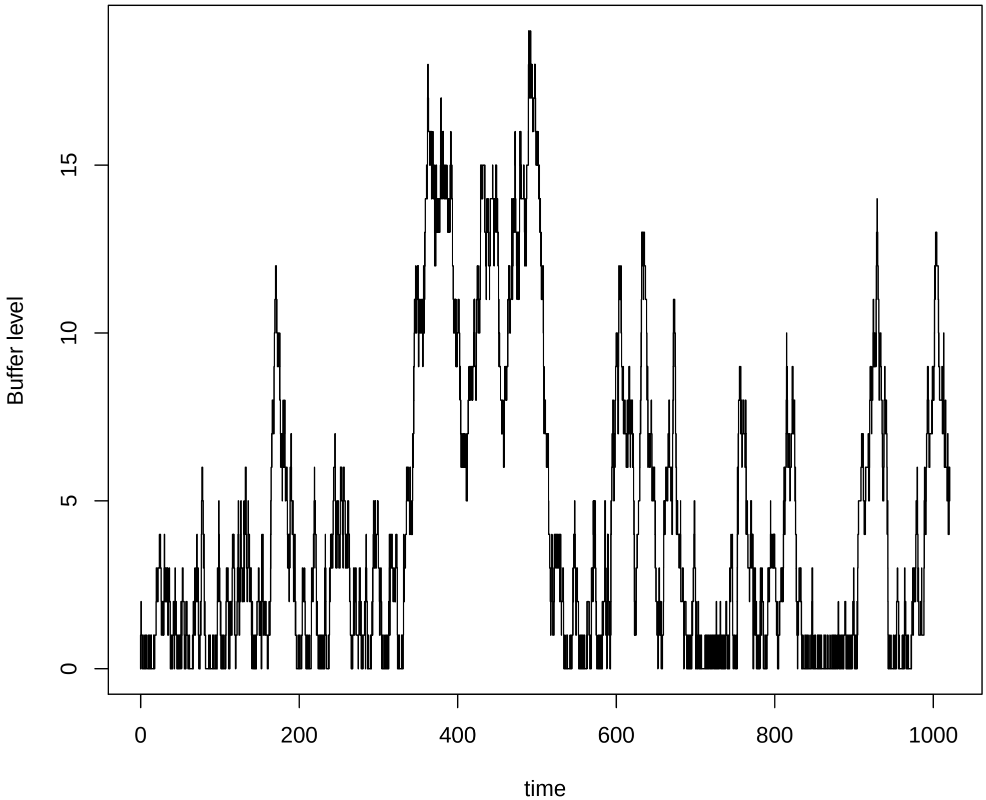
so conditional $W|W > 0$ is exponential with parameter $(1-\rho)\mu$!

How does WIP behave over time?



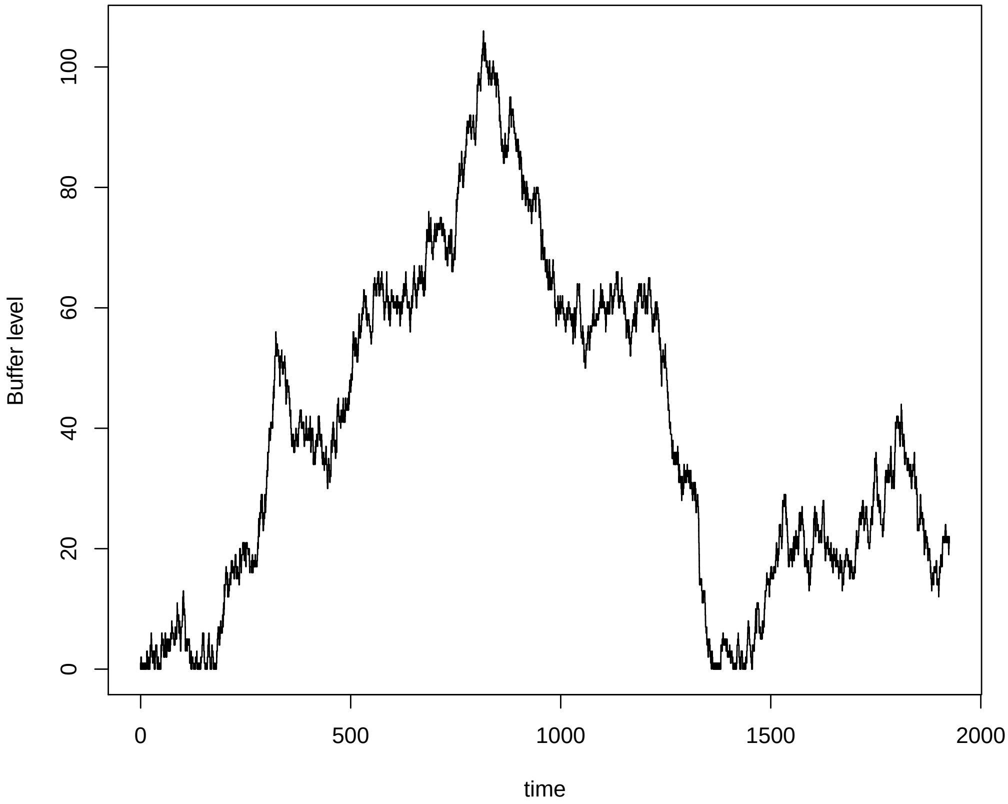
Exponential model, $\lambda = 1.0$, $\rho = 0.5$

How does WIP behave over time?



Exponential model, $\lambda = 1.0$, $\rho = 0.9$

How does WIP behave over time?



Exponential model, $\lambda = 1.0$, $\rho = 0.95$

Single server: general service

- Poisson arrivals with rate λ
- General service times B with distribution $F_B(\cdot)$
- Stability: $\rho = \lambda E(B) < 1$
- Single server and FCFS service

Then:

$$E(W) = E(Q^a)E(B) + \rho E(R) = E(Q)E(B) + \rho E(R)$$

where Q (Q^a) is number in queue (on arrival) and R is residual service time,

$$E(R) = \frac{E(B^2)}{2E(B)} = \frac{1}{2}\, E(B)\, (1 + c_B^2).$$

So with Little's law $E(Q) = \lambda E(W)$, we get...

Single server: general service

- Poisson arrivals with rate λ
- General service times B with distribution $F_B(\cdot)$
- Stability: $\rho = \lambda E(B) < 1$
- Single server and FCFS service

Then:

$$E(W) = \frac{\rho E(R)}{1-\rho}$$

where R is residual service time,

$$E(R) = \frac{E(B^2)}{2E(B)} = \frac{1}{2} E(B) \left(1 + c_B^2\right)$$

so

$$E(W) = \frac{\rho}{1-\rho} \frac{1}{2} E(B) \left(1 + c_B^2\right)$$

Single server: residual service

Probability of randomly selected service X of size x is proportional to its size x and $f_B(x)dx$, which is number of services of size x:

$$P(x < X < x + dx) = f_X(x)dx = Cxf_B(x)dx$$

where C is normalizing constant, so $C = 1/\int_{x=0}^{\infty} xf_B(x)dx = 1/E(B)$ and

$$f_X(x) = \frac{xf_B(x)}{E(B)}$$

The mean of randomly selected service is

$$E(X) = \int_{x=0}^{\infty} xf_X(x)dx = \frac{1}{E(B)}\int_{x=0}^{\infty} x^2 f_B(x)dx = \frac{E(B^2)}{E(B)}$$

On average, in the middle of randomly selected service, so

$$E(R) = \frac{E(X)}{2} = \frac{E(B^2)}{2E(B)}$$

Single server: general

- General inter-arrival times A with distribution $F_A(\cdot)$, mean $E(A)$, sd $\sigma(A)$
- General service times B with distribution $F_B(\cdot)$, mean $E(B)$, sd $\sigma(B)$
- Stability: $\rho = E(B)/E(A) < 1$
- Single server and FCFS service

Then:

$$E(W) \approx \frac{\rho}{1-\rho}\,\frac{1}{2}\,E(B)\,(c_A^2 + c_B^2)$$

where c_A and c_B are coefficients of variation of A and B:

$$c_A = \frac{\sigma(A)}{E(A)}, \quad c_B = \frac{\sigma(B)}{E(B)}.$$

Single server: general

$$E(W) \approx \frac{\rho}{1-\rho}\,\frac{1}{2}\,E(B)\,(c_A^2 + c_B^2) \qquad (1)$$

Lessons:

- As ρ tends to 1 then $E(W)$ tends to ∞
- As ρ tends to 1 then approximation (1) becomes *exact*: the relative error tends to 0
- As ρ tends to 1 then waiting time distribution becomes exponential
- Summary: As system operates close to its maximum capacity, waiting times are long and exponential with mean (1)
- Insensitivity: Mean waiting time only depends on mean and standard deviation of inter-arrival times and service times!

Single server: output process

Inter-departure times D:

- Conservation of flow gives $E(D) = E(A)$ or output rate = input rate
- Squared coefficient of variation $c_D^2 \approx (1 - \rho^2)c_A^2 + \rho^2 c_B^2$
- This makes sense since:
  - If $\rho \approx 1$, then server nearly always busy, so

$$c_D \approx c_B$$

  - If $\rho \approx 0$, then $E(B)$ is very small compared to $E(A)$, so

$$c_D \approx c_A$$

Multi server: exponential

- Poisson arrivals with rate λ
- Exponential service times with mean $1/\mu$
- Stability: $\lambda < c\mu$ or $\rho = \lambda/(c\mu) < 1$
- c parallel servers and FCFS service

Then:

$$p_k = P(k \text{ in system}) = \begin{cases} \frac{(c\rho)^k}{k!} p_0 & k = 0, 1, \ldots c-1, \\ \rho^{k-c} \frac{(c\rho)^c}{c!} p_0 & k = c, c+1, \ldots \end{cases}$$

where

$$\frac{1}{p_0} = \sum_{k=0}^{c-1} \frac{(c\rho)^k}{k!} + \frac{(c\rho)^c}{c!} \frac{1}{1-\rho}.$$

Multi server: exponential

- Poisson arrivals with rate λ
- Exponential service times with mean $1/\mu$
- Stability: $\lambda < c\mu$ or $\rho = \lambda/(c\mu) < 1$
- c parallel servers and FCFS service

Then:

- $E(W) = E(Q)\frac{1}{c\mu} + \Pi_W \frac{1}{c\mu}$, so with Little's law, $E(W) = \frac{\Pi_W}{1-\rho}\,\frac{1}{c\mu}$
- Π_W is probability of waiting,

$$\begin{aligned}\Pi_W &= P(W > 0) \\ &= p_c + p_{c+1} + \cdots = \frac{(c\rho)^c}{c!}\left(\frac{(c\rho)^c}{c!} + (1-\rho)\sum_{n=0}^{c-1}\frac{(c\rho)^n}{n!}\right)^{-1}\end{aligned}$$

- $P(W > t) = \Pi_W e^{-c\mu(1-\rho)t}$, $t \geq 0$ (nearly Exponential)

Multi server: exponential

- Let a_k be arrival distribution, then $a_k = p_k$ by PASTA
- Let d_k be departure distribution, then $d_k = a_k$ $(= p_k)$
- Output process is again Poisson!

Multi server: general service

- Poisson arrivals with rate λ
- General service times B with distribution $F_B(\cdot)$
- Stability: $\rho = \lambda E(B)/c < 1$
- c parallel servers and FCFS service

Then:

$$E(W) \approx \frac{\Pi_W}{1-\rho}\frac{E(R)}{c}$$

where Π_W is probability of waiting in corresponding exponential system, so

$$E(W) \approx \frac{\Pi_W}{1-\rho}\frac{1}{2}\frac{E(B)}{c}(1+c_B^2).$$

- Π_W is fairly insensitive to service time distribution;
- Corresponding system means with same mean service times.

Multi server: general

- General inter-arrival times A with distribution $F_A(\cdot)$, mean $E(A)$, sd $\sigma(A)$
- General service times B with distribution $F_B(\cdot)$, mean $E(B)$, sd $\sigma(B)$
- Stability: $\rho = E(B)/(cE(A)) < 1$
- c parallel servers and FCFS service

Then:

$$E(W) \approx \frac{\Pi_W}{1-\rho}\,\frac{1}{2}\,\frac{E(B)}{c}\,(c_A^2 + c_B^2)$$

where c_A and c_B are coefficients of variation of A and B

Multi server: output process

Inter-departure times D:

- Conservation of flow gives $E(D) = E(A)$ or output rate = input rate
- Coefficient of variation $c_D^2 \approx 1 + (1-\rho^2)(c_A^2 - 1) + \rho^2(c_B^2 - 1)/\sqrt{c}$