Stochastic Models of Manufacturing Systems

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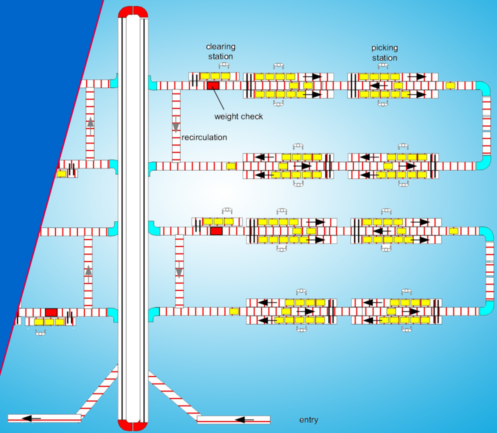

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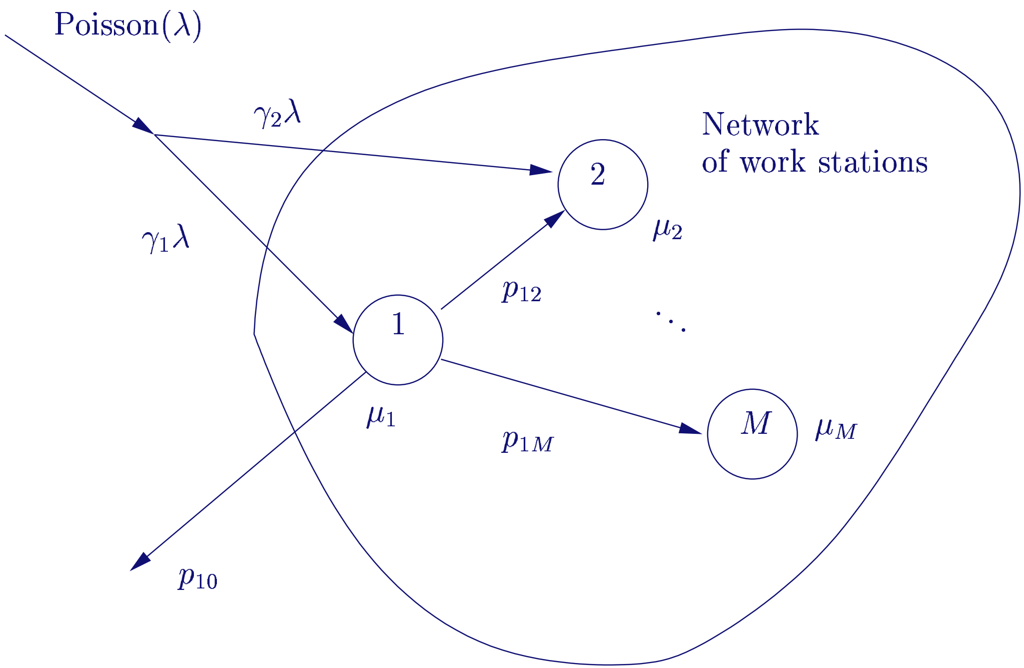
Tuesday June 9

Exponential open job shops

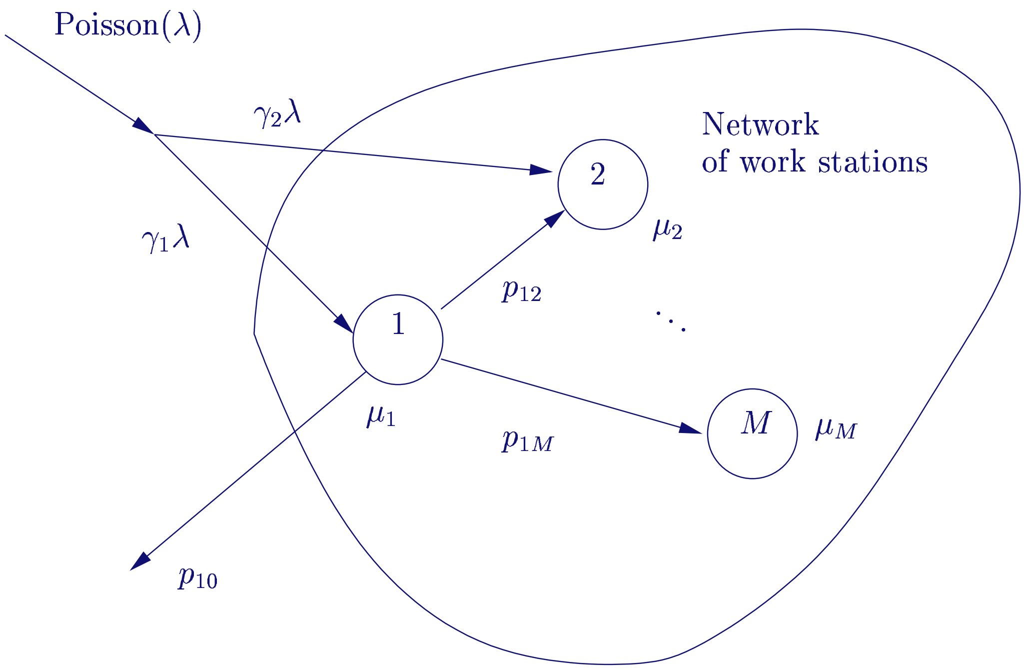
- Workstations $1, \ldots, M$
- Workstation m has c_m parallel identical machines
- Jobs arrive according to Poisson stream with rate λ
- Arriving job joins workstation m with probabilty γ_m
- Processing times in workstation m are exponential with rate μ_m
- Processing order is FCFS
- Buffers are unlimited
- Markovian routing:
  job moves from workstation m to n with probability p_{mn} and leaves system with probability p_{m0}

This network is also called Open Jackson network

Exponential open job shops



Exponential open job shops



Network capacity

Let v_m be average number of visits (of a job) to work station m

$$v_m = \gamma_m + \sum_{n=1}^{M} v_n p_{nm}, \quad m = 1, \ldots, M.$$

Equations have unique solution for $v_1, \ldots, v_M$.

Then λv_m is total number of visits per time unit to work station m.

Bottleneck station: station with the highest load

$$\max_{1 \le m \le M} \lambda v_m \cdot \frac{1}{c_m \mu_m}$$

Stability: For all m,

$$\lambda v_m \cdot \frac{1}{c_m \mu_m} < 1.$$

Exponential single server network

States of network $\underline{k} = (k_1, \ldots, k_M)$ where k_m is number of jobs in station m

State probabilities $p(k_1, k_2, \ldots, k_M)$ satisfy balance equations ($c_m = 1$)

Flow out of $\underline{k}$ = Flow into $\underline{k}$

$$
\begin{aligned}
p(\underline{k})\left(\lambda + \sum_{m=1}^{M} \mu_m \epsilon(k_m)\right) &= \sum_{m=1}^{M} p(\underline{k} + \underline{e}_m)\mu_m p_{m0} \\
&+ \sum_{n=1}^{M}\sum_{m=1}^{M} p(\underline{k} + \underline{e}_n - \underline{e}_m)\mu_n p_{nm}\epsilon(k_m) \\
&+ \sum_{m=1}^{M} p(\underline{k} - \underline{e}_m)\lambda\gamma_m\epsilon(k_m).
\end{aligned}
$$

where $\underline{e}_m = (0, \ldots, 1, \ldots, 0)$ with 1 at place m and $\epsilon(k) = \begin{cases} 1 & \text{if } k > 0 \\ 0 & \text{else} \end{cases}$

Exponential single server network

Product form solution “Jackson’s miracle”

$$p(\underline{k}) = p_1(k_1)p_2(k_2)\cdots p_M(k_M),$$

where

$$p_m(k_m) = (1-\rho_m)\rho_m^{k_m}, \quad k_m = 0, 1, \ldots$$

and

$$\rho_m = \frac{\lambda_m}{\mu_m}$$

with $\lambda_m = v_m\lambda$ total arrival rate to workstation m

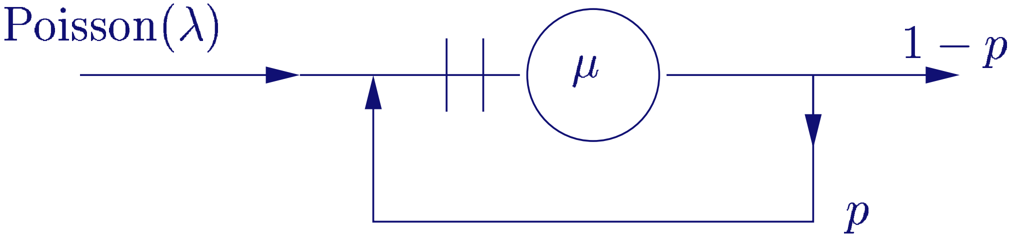
This is just the product of $M/M/1$ solutions!

Exponential single server network

Surprising result:

- Marginal distribution $p_m(\cdot)$ is exactly the same as distribution of $M/M/1$ with arrival rate λ_m and service rate μ_m
- Inflow to workstation m is in general not Poisson!

Example: $\mu = 1/\epsilon$, $p = 1 - \epsilon$ (so $\mu(1-p) = 1$), $\lambda \ll 1$



Arrival pattern:  (So not Poisson at all!)

Question: What is the solution for an exponential multi-server network?

Exponential open job shops

- Queue lengths at workstations are independent (if you take a snapshot)!
- Inflow to workstation m is in general not Poisson! But you can “do as if”
- Infinite server station $c_m = \infty$

$$p_m(k_m) = e^{-\rho_m} \frac{\rho_m^{k_m}}{k_m!} \quad \text{(Poisson distribution)}$$

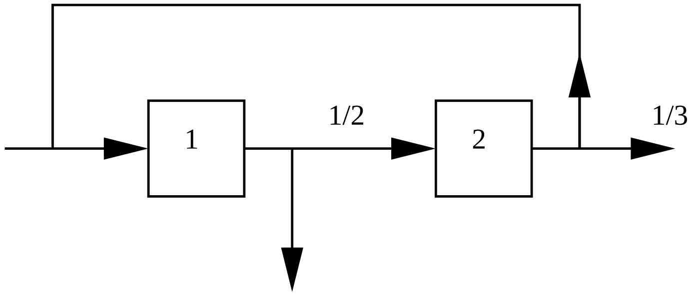
where $\rho_m = \lambda_m / \mu_m$

- Distribution for $c_m = \infty$ also valid for general service time distribution!
- Infinite server stations useful to describe transportation delay
- Product form result also valid for fixed route $C_1, C_2, \ldots, C_n$ in which case

$$\lambda_m = \lambda \sum_{i=1}^{n} \mathbf{1}[C_i = m]$$

where $\mathbf{1}[C_i = m] = 1$ if $C_i = m$ and 0 otherwise (C_i is workstation i)

Example

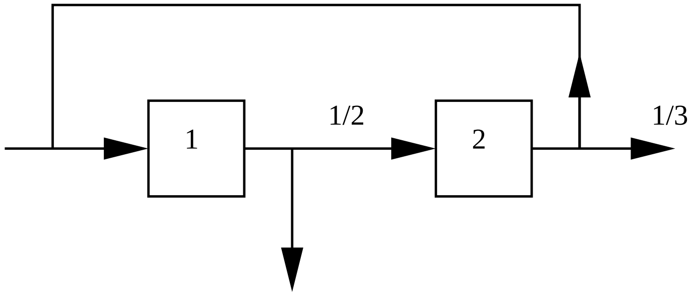


- $M = 2$ workstations
- $c_1 = c_2 = 1$ (single server network)
- Arriving jobs join workstation 1 ($\gamma_1 = 1$)
- Markovian routing: $p_{12} = \frac{1}{2}$, $p_{21} = \frac{2}{3}$

Then:

- $\lambda_1 = \frac{3}{2}\lambda$, $\lambda_2 = \frac{3}{4}\lambda$
- Stability: $\lambda_1 < \mu_1$, $\lambda_2 < u_2$, so $\lambda < \min\{\frac{2}{3}\mu_1, \frac{4}{3}\mu_2\}$
- $E(S_1) = \frac{1}{\mu_1 - \lambda_1}$, $E(S_2) = \frac{1}{\mu_2 - \lambda_2}$, $E(S) = \frac{3}{2}E(S_1) + \frac{3}{4}E(S_2)$

Example



- $M = 2$ workstations
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Questions: What is the mean total flow time in case of

- Fixed routing: $1, 2, 1$?
- Fixed transportation delays T in between workstations?
- 3 types of jobs: 50% with routing 1, 2, 1; 25% with routing 1; 25% with routing 1, 2

General open job shops

- Workstations $1, \ldots, M$
- Workstation m has c_m parallel identical machines
- Arrival process with general inter-arrival times A, mean $\frac{1}{\lambda}$ and scv c_A^2
- Arriving job joins workstation m with probabilty γ_m
- General processing times B_m in station m with mean $E(B_m)$ and scv $c_{B_m}^2$
- Processing order is FCFS
- Buffers are unlimited
- Markovian routing:
  job moves from workstation m to n with probability p_{mn} and leaves system with probability p_{m0}

Network capacity

Let v_m be average number of visits to work station m

$$v_m = \gamma_m + \sum_{n=1}^{M} v_n p_{nm}, \quad m = 1, \ldots, M.$$

Equations have unique solution for $v_1, \ldots, v_M$

Bottleneck station: station with the highest load

$$\max_{1 \leq m \leq M} \lambda v_m \cdot \frac{1}{c_m \mu_m}$$

where $\mu_m = 1/E(B_m)$ is processing rate of station m

Stability: For all m,

$$\lambda v_m \cdot \frac{1}{c_m \mu_m} < 1.$$

General open job shops

Lesson learned from Jackson networks:

- Each station can be analyzed in isolation with appropriate arrival process
- These results can be combined to produce overall performance

Approach for general open job shops:

- Model each work station m as $G/G/c$ with $c = c_m$, $B = B_m$ and $A = A_m$
- A_m is inter-arrival time at workstation m with

$$E(A_m) = 1/\lambda_m \quad (\lambda_m = v_m \lambda)$$

  and an appropriate scv $c^2_{A_m}$
- Overall performance:

$$p(k_1, k_2, \ldots, k_M) \approx p_1(k_1)\, p_2(k_2) \cdots p_M(k_M)$$

  where $p_m(k_m)$ be (approximate) queue length distribution of this $G/G/c_m$

General open job shops

What is an appropriate $c^2_{A_m}$?

Estimate $c^2_{A_m}$ using:

- Merging of streams: If two streams with rates λ_1 and λ_2 and scv $c^2_{A_1}$ and $c^2_{A_2}$ are merged, then the resulting stream has rate $\lambda_1 + \lambda_2$ and its scv can be approximated by

$$c^2_A \approx \frac{\lambda_1}{\lambda_1 + \lambda_2} c^2_{A_1} + \frac{\lambda_2}{\lambda_1 + \lambda_2} c^2_{A_2}$$

- Random splitting of stream: If arriving jobs are randomly split with probability p from a stream with rate λ and scv c^2, then the resulting stream has rate $p\lambda$ and

$$c^2_A = pc^2 + 1 - p$$

General open job shops

(Fixed-point) Equations for $c^2_{A_m}$:

$$c^2_{A_m} = \frac{\lambda\gamma_m}{\lambda_m}\Big[\gamma_m c^2_A + (1-\gamma_m)\Big] + \sum_{n=1}^{M}\frac{\lambda_n p_{nm}}{\lambda_m}\Big[p_{nm}c^2_{D_n} + (1-p_{nm})\Big]$$

for all $m = 1, \ldots, M$, where

$$c^2_{D_n} = 1 + (1-\rho^2_n)(c^2_{A_n} - 1) + \rho^2_n(c^2_{B_n} - 1)/\sqrt{c_n}$$

and $\rho_n = \lambda_n E(B_n)/c_n$

Observation:

For “large randomly routed networks” the arrival process at each station can be approximated by a “random” process, thus Poisson process ($c^2_{A_m} = 1$)

General multi-class open job shops

- Workstations $1, \ldots, M$
- Workstation m has c_m parallel identical machines
- Job types $1, \ldots, R$
- Jobs of type r arrive at rate Λ_r and scv of inter-arrival times is c_r^2
- Jobs of type r require n_r operations, labeled $1, 2, \ldots, n_r$
- Jobs of type r follow fixed routing: $C_{1r}, C_{2r}, \ldots\ldots, C_{n_r r}$
  where C_{ir} is workstation for operation i, with processing time B_{ir}
- Processing order is FCFS
- Buffers are unlimited

Question: What is mean total flow time $E(S_r)$ of type r job?

General multi-class open job shops

Approximation approach for large random network:

- Model workstation m as $M/G/c$ with $c = c_m$
- Arrival rate at station m

$$\lambda_m = \sum_{r=1}^{R} \Lambda_r \sum_{i=1}^{n_r} \mathbf{1}[C_{ir} = m]$$

- Service time B_m of arbitrary job

$$E(B_m) = \sum_{r=1}^{R} \frac{\Lambda_r}{\lambda_m} \sum_{i=1}^{n_r} \mathbf{1}[C_{ir} = m] E(B_{ir})$$

$$E(B_m^2) = \sum_{r=1}^{R} \frac{\Lambda_r}{\lambda_m} \sum_{i=1}^{n_r} \frac{\Lambda_r}{\lambda_m} \mathbf{1}[C_{ir} = m] E(B_{ir}^2)$$

General multi-class open job shops

Approximation approach for large random network:

- Mean waiting time (which does not depend on job type!)

$$E(W_m) = \frac{\Pi_W}{1-\rho_m} \frac{E(R_m)}{c_m}$$

where $\rho_m = \lambda_m E(B_m)/c_m$, $E(R_m) = \frac{1}{2}E(B_m^2)/E(B_m)$ and
Π_W is probability of waiting in $M(\lambda_m)/M(\mu_m)/c_m$ with $\mu_m = 1/E(B_m)$

- Mean flow time of operation i of type r job

$$E(S_{ir}) = \sum_{m=1}^{M} E(W_m)\mathbf{1}[C_{ir} = m] + E(B_{ir})$$

- Mean total flow time of type r job

$$E(S_r) = \sum_{i=1}^{n_r} E(S_{ir})$$

Example

Production system with two single-machine work stations and two job types

r	Λ_r (jobs/hour)	C_{ir}	$E(B_{ir})$ (min)	$\sigma(B_{ir})$	$E(B_{ir}^2)$
1	3	1,2,1	10,5,6	2,5,2	104,50,40
2	2	2	20	0	400

Thus processing characteristics of an arbitrary job in station 1 and 2

m	λ_m (jobs/hour)	$E(B_m)$ (min)	$E(B_m^2)$	ρ_m
1	6	8	72	0.80
2	5	11	190	0.92

Hence

$$E(W_1) = 18(\text{min}), \quad E(W_2) = 99.3(\text{min})$$

so

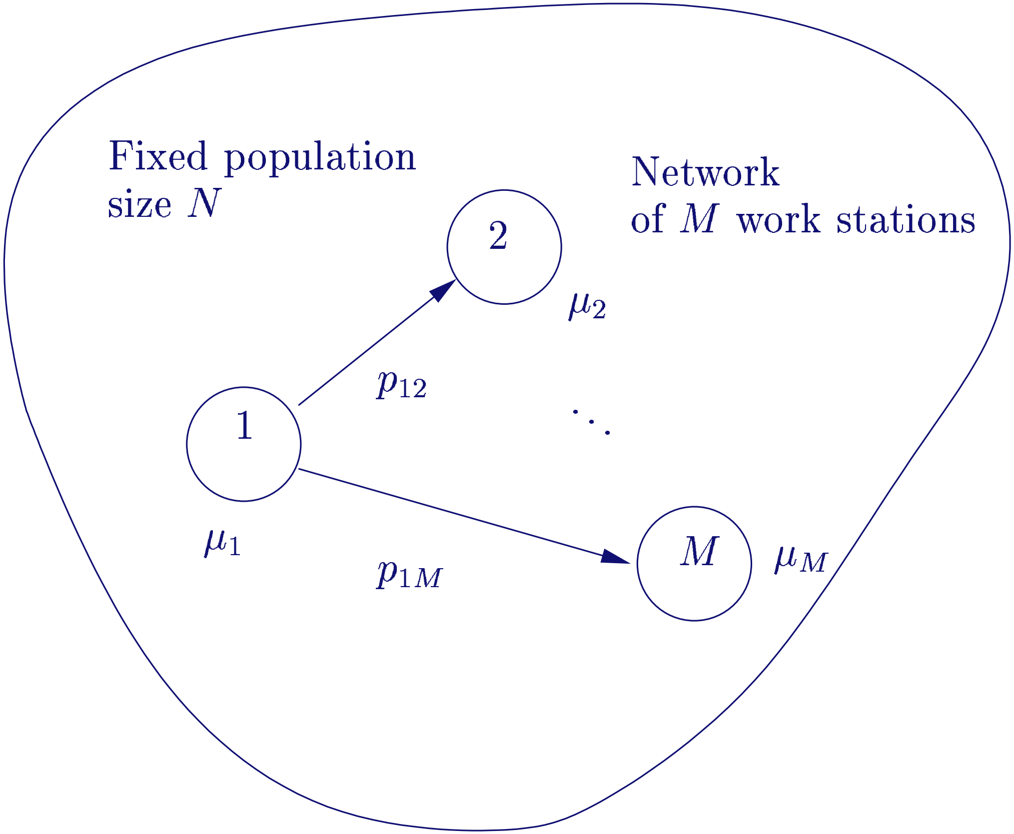
$$E(S_1) = 18 + 10 + 99.3 + 5 + 18 + 6 = 156.3(\text{min}), \quad E(S_2) = 119.3(\text{min})$$

Exponential closed networks

- Workstations $1, \ldots, M$
- Workstation m has c_m parallel identical machines
- N circulating jobs (N is the population size)
- Processing times in workstation m are exponential with rate μ_m
- Processing order is FCFS
- Buffers are unlimited
- Markovian routing:
  job moves from workstation m to n with probability p_{mn}

This network is also called Closed Jackson network

Exponential closed networks



Exponential single server network

States of network $\underline{k} = (k_1, \ldots, k_M)$ where k_m is number of jobs in station m

Note that

$$\sum_{m=1}^{M} k_m = N$$

so there are $\binom{N+M-1}{M-1}$ states!

State probabilities $p(k_1, k_2, \ldots, k_M)$ satisfy balance equations ($c_m = 1$)

$$\text{Flow out of } \underline{k} \quad = \quad \text{Flow into } \underline{k}$$

$$p(\underline{k}) \sum_{m=1}^{M} \mu_m \epsilon(k_m) = \sum_{n=1}^{M} \sum_{m=1}^{M} p(\underline{k} + \underline{e}_n - \underline{e}_m) \mu_n p_{nm} \epsilon(k_m)$$

where $\underline{e}_m = (0, \ldots, 1, \ldots, 0)$ with 1 at place m and $\epsilon(k) = \begin{cases} 1 & \text{if } k > 0 \\ 0 & \text{else} \end{cases}$

Exponential single server network

Product form solution “Jackson’s miracle”

$$p(\underline{k}) = C p_1(k_1) p_2(k_2) \cdots p_M(k_M),$$

where C is normalizing constant and

$$p_m(k_m) = \left(\frac{v_m}{\mu_m}\right)^{k_m}, \quad k_m = 0, 1, \ldots$$

with v_m the “arrival rate” to workstation m

This is again the product of $M/M/1$ solutions!

But: What is v_m?

Exponential single server network

v_m is the relative arrival rate or visiting frequency to m, satisfying

$$v_m = \sum_{n=1}^{M} v_n p_{nm}, \quad m = 1, \ldots, M$$

Remarks:

- Equations above determine v_m's up to a multiplicative constant
- Set $v_1 = 1$, then v_m is the expected number of visits to m in between two successive visits to station 1
- Although $p(\underline{k})$ is again a product, the queues at stations are dependent!
- How to compute C?

Normalizing constant

Let

$$C(m,n) = \sum_{\substack{k_1,\ldots,k_m \geq 0 \\ \sum_{i=1}^m k_i = n}} \left(\frac{v_1}{\mu_1}\right)^{k_1} \left(\frac{v_2}{\mu_2}\right)^{k_2} \cdots \left(\frac{v_m}{\mu_m}\right)^{k_m}.$$

Then $C = 1/C(M,N)$

Recursion:

$$C(m,n) = C(m-1,n) + \frac{v_m}{\mu_m} C(m,n-1)$$

with initial conditions

$$C(0,n) = 0, \quad n = 1,\ldots,N, \quad C(m,0) = 1, \quad m = 1,\ldots,M,$$

Absolute arrival rate

- What is the real arrival rate λ_m?
  Note that

$$\begin{aligned} \text{input rate to station } M &= \text{output rate from station } M \\ \lambda_M &= v_m \frac{C(M, N-1)}{C(M, N)} \end{aligned}$$

  and $\lambda_m / \lambda_M = v_m / v_M$

- What is mean number $E(L_M)$ in station M?

$$E(L_M) = \frac{\sum_{k_M=0}^{N} k_M \left(\frac{v_M}{\mu_M}\right)^{k_M} C(M-1, N-k_M)}{C(M, N)}$$

- What is expected cycle time $E(C)$ between two visits to station 1?
  By Little's law

$$\lambda_1 E(C) = N$$