Analysis of Manufacturing Systems 4AB00

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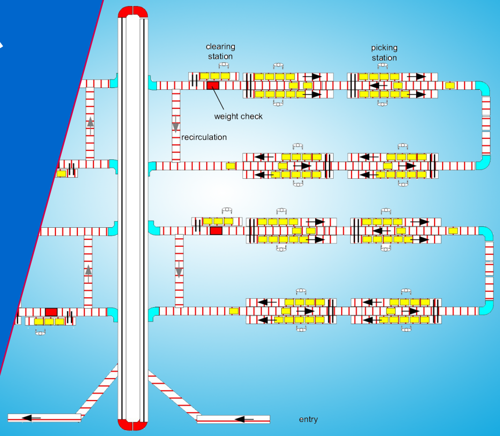

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Lecture notes on Analysis of Manufacturing Systems

- Chapter 3: 3.1-3.4
- Chapter 6
- Chapter 7

Recap: Central limit theorem

$X_1, X_2, \dots$ are independent random variables with the same distribution, and

$$\mu = E(X), \quad \sigma = \sigma(X)$$

Then $X_1 + \cdots + X_n$ has approximately a normal distribution when n is large

Recap: Joint random variables

- Joint probability distribution function of continuous variables X and Y

$$P(X \leq a, Y \leq b) = \int_{x=-\infty}^{a} \int_{y=-\infty}^{b} f(x, y) dx dy$$

- Joint density $f(x, y)$ satisfies

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} P(X \leq x, Y \leq y)$$

- Interpretation of joint density: for small $\Delta > 0$

$$P(x < X \leq x + \Delta, y < Y \leq y + \Delta) \approx f(x, y)\Delta^2$$

- X and Y are independent if

$$f(x, y) = f_X(x) f_Y(y) \quad \text{for all } x, y.$$

Recap: Correlation

- Covariance of X and Y is

$$
\begin{aligned}
\text{cov}(X,Y) &= E[(X-E(X))(Y-E(Y))] \\
&= E(XY) - E(X)E(Y)
\end{aligned}
$$

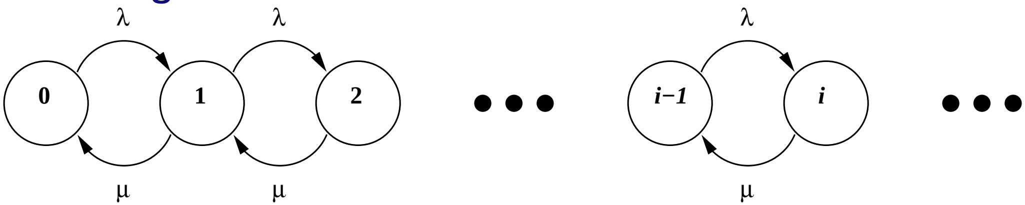
- If X and Y are independent, then $\text{cov}(X,Y) = 0$
- Correlation coefficient of X and Y is “measure of dependence”

$$
\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sigma(X)\sigma(Y)}
$$

- Property: $-1 \leq \rho(X,Y) \leq 1$
- X and Y are uncorrelated if $\rho(X,Y) = 0$
- Independent random variables are uncorrelated

Recap: Exponential machine model

- Poisson arrivals: inter-arrival time is exponential with rate λ
- Process time is exponential with rate μ
- Infinite capacity buffer
- p_i is long-run probability (or fraction of time) of finding i jobs in the system
- Flow diagram



- Balance equations: flow from state i to $i-1$ = flow from state $i-1$ to i
- Balance equations: $p_i\mu = p_{i-1}\lambda$ and thus

$$p_i = p_{i-1}\frac{\lambda}{\mu} = p_{i-2}\left(\frac{\lambda}{\mu}\right)^2 = \cdots = p_0\left(\frac{\lambda}{\mu}\right)^i$$

- p_0 is probability that machine is idle, so $p_0 = 1 - \frac{\lambda}{\mu}$

Recap: Exponential machine model

- Poisson arrivals: inter-arrival time is exponential with rate λ
- Process time is exponential with rate μ
- Geometric distribution of number in system

$$p_i = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^i$$

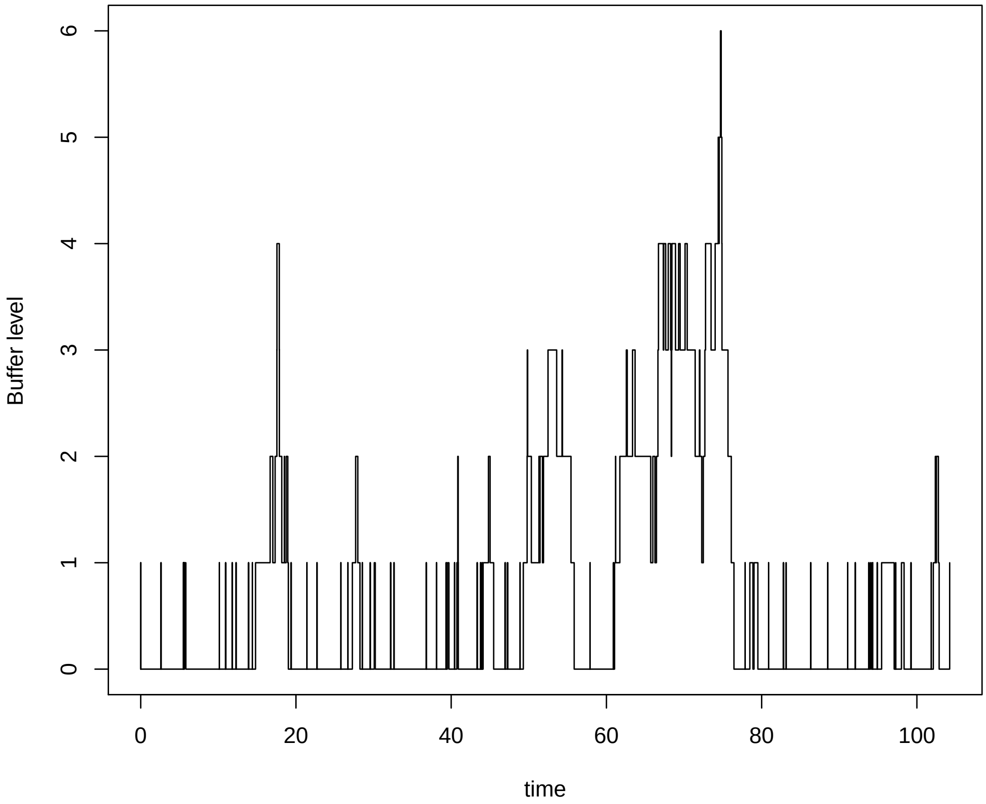
- Utilization $u = 1 - p_0 = \frac{\lambda}{\mu}$
- Mean Work-In-Process (WIP) level

$$w = \sum_{i=0}^{\infty} i p_i = \frac{\lambda/\mu}{1 - \lambda/\mu} = \frac{u}{1-u}$$

- Mean Flow time

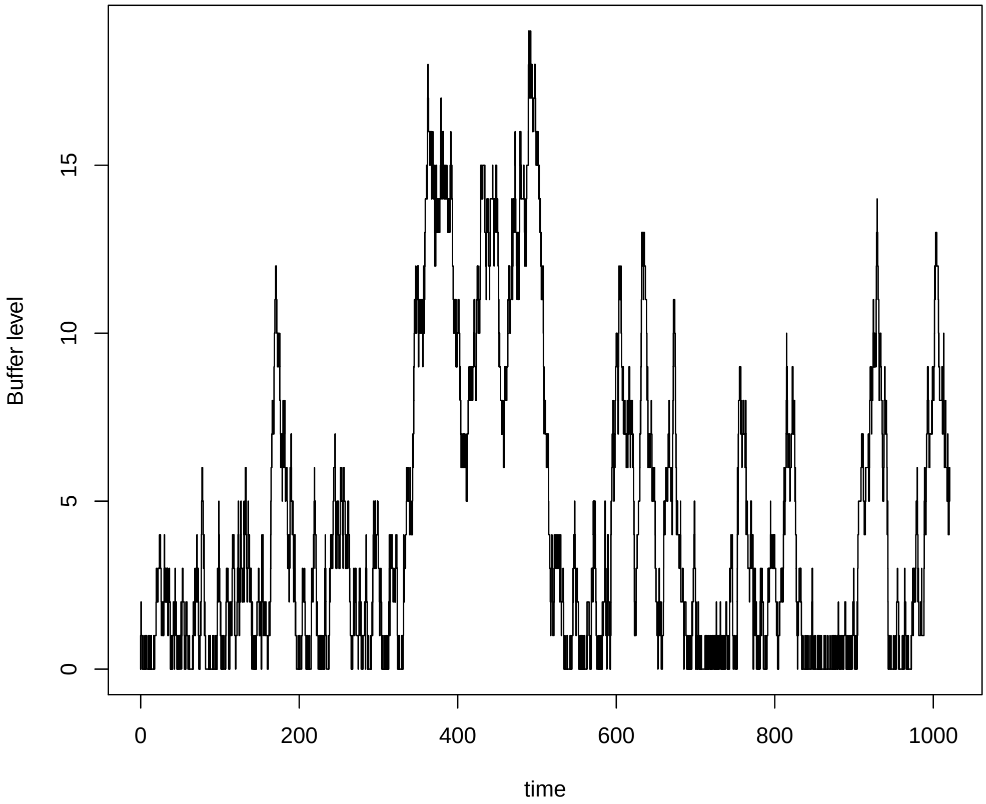
$$\varphi = \frac{w}{\lambda} = \frac{1/\mu}{1 - \lambda/\mu} = \frac{1/\mu}{1-u}$$

How does WIP behave over time?



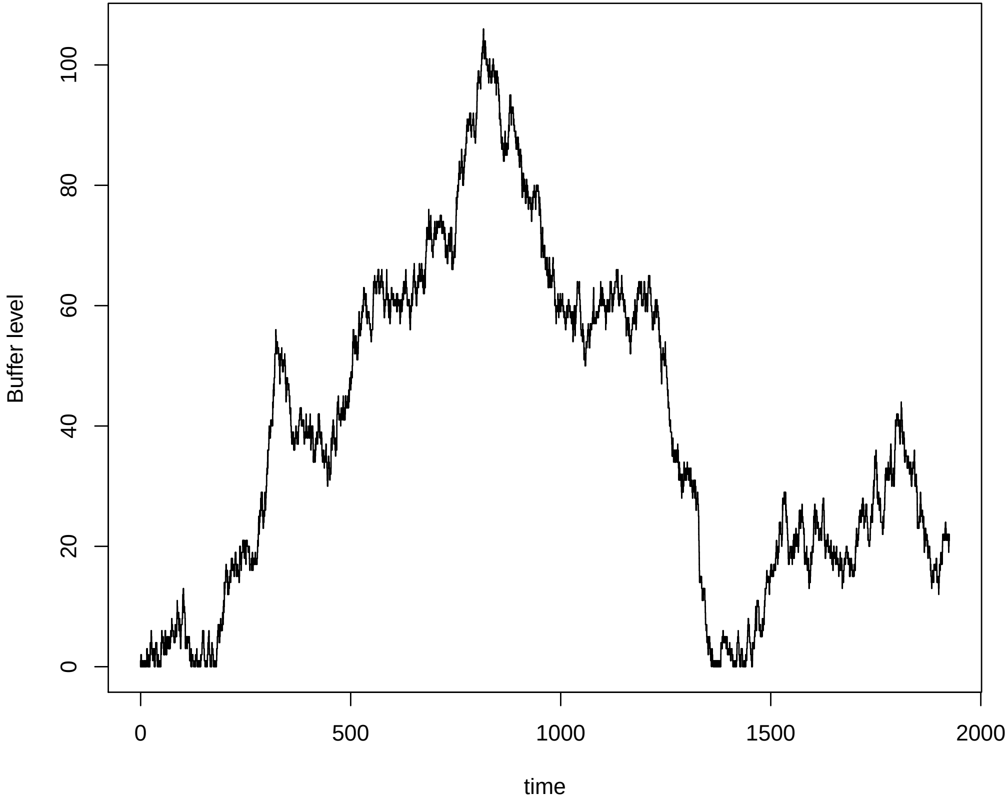
Exponential machine model, $t_a = 1.0$, $t_e = 0.5$, $u = 0.5$

How does WIP behave over time?



Exponential machine model, $t_a = 1.0$, $t_e = 0.9$, $u = 0.9$

How does WIP behave over time?



Exponential machine model, $t_a = 1.0$, $t_e = 0.95$, $u = 0.95$

Recap: Mean flow time

- Inter-arrival time is general with mean t_a and standard deviation σ_a
- Process time is general with mean t_e and standard deviation σ_e
- Infinite capacity buffer
- Single machine
- Processing in order of arrival

```
model real GIBME(real ta, te; int n):
    chan lot a, b;
    run G(a, uniform(0.0,2.0*ta)), B(a, b),
        M(b, c, uniform(0.0,2.0*te)), E(c, n)
end
```

Recap: Mean flow time

- Inter-arrival time is exponential with mean t_a
- Process time is exponential with mean t_e
- Infinite capacity buffer
- Single machine
- Processing in order of arrival

Exact result

$$\varphi = \left(\frac{u}{1-u} + 1 \right) \cdot t_e$$

with

- $u = t_e / t_a$

Recap: Mean flow time

- Inter-arrival time is exponential with mean t_a
- Process time is general with mean t_e and standard deviation σ_e
- Infinite capacity buffer
- Single machine
- Processing in order of arrival

Exact result

$$\varphi = \left(\gamma \cdot \frac{u}{1-u} + 1\right) \cdot t_e$$

with

- $u = t_e / t_a$
- $\gamma = \frac{1}{2} \cdot (1 + c_e^2)$
- $c_e = \sigma_e / t_e$

Recap: Mean flow time

- Inter-arrival time is general with mean t_a and standard deviation σ_a
- Process time is general with mean t_e and standard deviation σ_e
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Approximation

$$\varphi \approx \left(\gamma \cdot \frac{u}{1-u} + 1 \right) \cdot t_e$$

with

- $u = t_e / t_a$
- $\gamma = \frac{1}{2} \cdot (c_a^2 + c_e^2)$
- $c_e = \sigma_e / t_e$
- $c_a = \sigma_a / t_a$

Single machine model

Example:

- Inter-arrival time is exponential with $\lambda = \frac{1}{4}$, so $t_a = 4$
- Process time is uniform on $(0, 6)$, so $t_e = 3$, $c_e^2 = \frac{1}{3}$

Question:

- Estimate for φ?

Answer:

$$\begin{aligned} t_a &= 4 \\ c_a^2 &= 1 \\ t_e &= 3 \\ c_e^2 &= \frac{1}{3} \\ u &= \frac{3}{4} \\ \varphi &= 9, \text{ and simulation gives } 9.05 \pm 0.07 \end{aligned}$$

Three machines in series

Example:

- Inter-arrival time is exponential with $\lambda = \frac{1}{4}$, so $t_a = 4$
- Process time is uniform on $(0, 6)$, so $t_e = 3$, $c_s^2 = \frac{1}{3}$ for each machine

Question:

- Estimate for φ?

```
model real Mline(real ta; int n):
    list(4) chan lot a, b;

    run G(a[0], exponential(ta)),
        unwind j in range(3):
            B(a[j], b[j]),
            M(b[j], a[j+1], uniform(0.0,6.0))
        end,
        E(a[3], n)
end
```

Three machines in series

Example:

- Inter-arrival time is exponential with $\lambda = \frac{1}{4}$, so $t_a = 4$
- Process time is uniform on $(0, 6)$, so $t_e = 3$, $c_e^2 = \frac{1}{3}$ for each machine

Question:

- Estimate for φ?

Answer:

- $\varphi = 22.5 \pm 0.4$ (by simulation)

Question:

- How to estimate φ analytically?

Note that output of a machine is input to down stream machine!

Departure process single machine

- Inter-arrival time is general with mean t_a and standard deviation σ_a
- Process time is general with mean t_e and standard deviation σ_e
- Single machine

Inter-departure time has mean t_d and standard deviation σ_d. What are t_d, σ_d?

- Output rate is input rate (conservation of flow)

$$t_d = t_a$$

- Coefficient of variation $c_d = \sigma_d / t_d$ is approximately equal to

$$c_d^2 \approx (1 - u^2) \cdot c_a^2 + u^2 \cdot c_e^2 \quad \text{(weighted average of } c_a^2 \text{ and } c_e^2\text{)}$$

with $c_a = \sigma_a / t_a$, $c_e = \sigma_e / t_e$ and $u = t_e / t_a$

Three machines in series

Example:

- Inter-arrival time is exponential with $\lambda = \frac{1}{4}$, so $t_a = 4$
- Process time is uniform on $(0, 6)$, so $t_e = 3$, $c_s^2 = \frac{1}{3}$ for each machine

Question:

- Estimate for φ?

Answer:

- $\varphi_1 = 9$, $u_1 = \frac{3}{4}$, $c_{d_1}^2 = \frac{7}{16} \cdot 1 + \frac{9}{16} \cdot \frac{1}{3} = \frac{5}{8}$
- $\varphi_2 \approx 7.3125$, $u_2 = \frac{3}{4}$, $c_{d_2}^2 = \frac{7}{16} \cdot \frac{5}{8} + \frac{9}{16} \cdot \frac{1}{3} = 0.4609$
- $\varphi_3 \approx 6.574$
- $\varphi = \varphi_1 + \varphi_2 + \varphi_3 \approx 22.89$

Question: How does φ grow as the length of this line tends to infinity?

Exponential machines in series

Example: m exponential machines in series

- Inter-arrival time is exponential with rate $\lambda = 1/t_a$
- Process time is exponential for machine i with rate $\mu_i = 1/t_i$
- Utilization $u_i = \lambda t_i < 1$

Mean flow time and mean number at machine i

$$\varphi_i = \frac{1/\mu_i}{1-u_i} = \frac{t_i}{1-u_i}, \quad \omega_i = \frac{u_i}{1-u_i}.$$

Total flow time and total work-in-process (WIP)

$$\varphi = \sum_{i=1}^{m} \varphi_i = \sum_{i=1}^{m} \frac{t_i}{1-u_i}, \quad \omega = \sum_{i=1}^{m} \omega_i = \sum_{i=1}^{m} \frac{u_i}{1-u_i}$$

Workload balancing

Question:

- How to allocate workload t among machines such that WIP is minimal?

$$\min \sum_{i=1}^{m} \frac{\lambda t_i}{1 - \lambda t_i}$$

subject to

$$\sum_{i=1}^{m} t_i = t$$

$$0 \leq \lambda t_i < 1, \quad i = 1, 2, \ldots, m$$

Note that $m > \lambda t$, otherwise no feasible allocation possible!

Workload balancing

Question:

- How to allocate workload t among machines such that WIP is minimal?

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subject to

$$\sum_{i=1}^{m} t_i = t$$

$$0 \leq \lambda t_i < 1, \quad i = 1, 2, \ldots, m$$

Note that $m > \lambda t$, otherwise no feasible allocation possible!

Answer: $t_i = t/m$, so balance the line!

Workload balancing

Question:

- What is impact of unbalance?

Workload balancing

Question:

- What is impact of unbalance?

Example: 4 machines in series, $\lambda = 1$, u is the average load per machine

$$u = \frac{1}{4}(u_1 + u_2 + u_3 + u_4).$$

u	t_1	t_2	t_3	t_4	ω_1	ω_2	ω_3	ω_4	ω
0.80	0.80	0.80	0.80	0.80	4.0	4.0	4.0	4.0	16.0
0.80	0.85	0.65	0.90	0.80	5.7	1.9	9.0	4.0	20.5
0.90	0.90	0.90	0.90	0.90	9.0	9.0	9.0	9.0	36.0
0.90	0.95	0.83	0.97	0.85	19.0	4.9	32.3	5.7	61.9
0.95	0.95	0.95	0.95	0.95	19.0	19.0	19.0	19.0	76.0
0.95	0.96	0.93	0.97	0.94	24.0	13.3	32.3	15.7	85.3

Balanced machines in series

- t_i is mean process time of machine i
- Balanced line, so $t_1 = \cdots = t_m$
- c_i^2 is squared coefficient of variation of process time of machine i
- Any order of machines is possible

Question:

- What is the order minimizing mean total flow time φ?

Balanced machines in series

- t_i is mean process time of machine i
- Balanced line, so $t_1 = \cdots = t_m$
- c_i^2 is squared coefficient of variation of process time of machine i
- Any order of machines is possible

Question:

- What is the order minimizing mean total flow time φ?

Answer:

$$c_1^2 \leq c_2^2 \leq \cdots \leq c_m^2,$$

so arrange machines in order of increasing processing variability!

Or: Reducing variability of process times closer to raw materials leads to greater improvement of line performance than reducing variability of process times closer to customers!

Balanced machines in series

Question:

- What is the impact of another order on the performance?

Balanced machines in series

Question:

- What is the impact of another order on the performance?

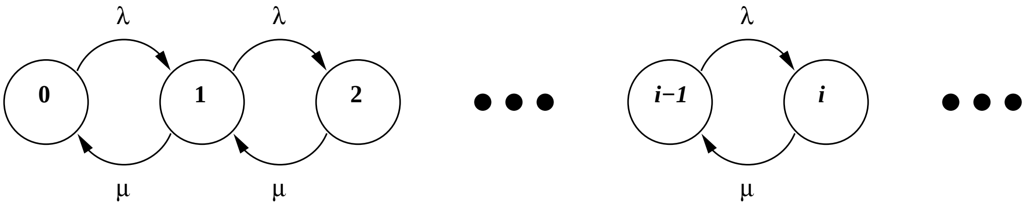
Example:

- Machines labeled $0, 1, 2$ with $t_i = 1$ and $c_i^2 = i$
- Poisson inflow with rate λ

order	$\lambda (= u)$	φ
2 1 0	0.80	12.8
0 1 2	0.80	10
2 1 0	0.90	30
0 1 2	0.90	22.3
2 1 0	0.95	64.8
0 1 2	0.95	47.2

Exponential single machine model

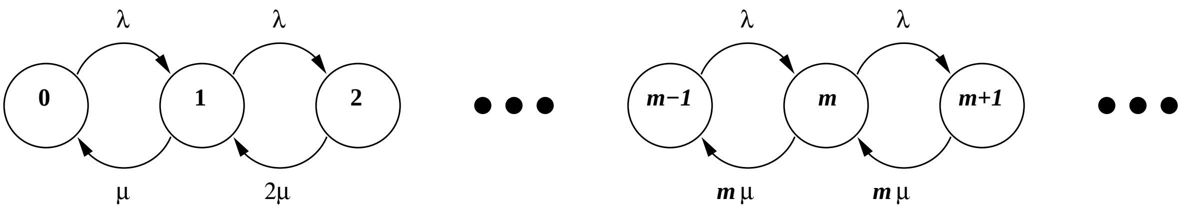
- Poisson arrivals: inter-arrival time is exponential with rate λ
- Process time is exponential with rate μ
- Single machine
- Stability $\lambda < \mu$
- p_i is long-run probability (or fraction of time) of finding i jobs in the system
- Flow diagram



- Balance equations: flow from state i to $i-1$ = flow from state $i-1$ to i

Exponential multi machine model

- Poisson arrivals: inter-arrival time is exponential with rate λ
- Process time is exponential with rate μ
- m parallel identical machines
- Stability $\lambda < m\mu$
- p_i is long-run probability (or fraction of time) of finding i jobs in the system
- Flow diagram



- Balance equations: flow from state i to $i-1$ = flow from state $i-1$ to i

Exponential multi machine model

- Balance equations:

$$i\mu p_i = \lambda p_{i-1}, \quad i \leq m$$

and thus

$$p_i = p_{i-1}\frac{\lambda}{i\mu} = p_{i-2}\frac{1}{i(i-1)}\left(\frac{\lambda}{\mu}\right)^2 = \cdots = p_0\frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i$$

- Balance equations:

$$m\mu p_i = \lambda p_{i-1}, \quad i > m$$

and thus

$$p_i = p_{i-1}\frac{\lambda}{m\mu} = p_{i-2}\left(\frac{\lambda}{m\mu}\right)^2 = \cdots = p_m\left(\frac{\lambda}{m\mu}\right)^{i-m}$$

$$p_m = p_0\frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^m$$

Exponential multi machine model

- p_0 is probability that system is empty
- p_0 follows from normalization $p_0 + p_1 + \cdots = 1$

$$\frac{1}{p_0} = \sum_{i=0}^{m-1} \frac{(\lambda/\mu)^i}{i!} + \frac{(\lambda/\mu)^m}{m!} \frac{1}{1 - \lambda/(m\mu)}$$

Exponential multi machine model

- Poisson arrivals: inter-arrival time is exponential with rate λ
- Process time is exponential with rate μ
- m parallel identical machines
- Stability $\lambda < m\mu$
- Utilization $u = \frac{\lambda}{m\mu}$
- Mean Flow time

$$\varphi = \frac{Q}{1-u}\frac{1}{m\mu} + \frac{1}{\mu}$$

- Q is probability that all machines are busy

$$\begin{aligned} Q &= p_m + p_{m+1} + \cdots \\ &= \frac{(mu)^m}{m!}\left(\frac{(mu)^m}{m!} + (1-u)\sum_{i=0}^{m-1}\frac{(mu)^i}{i!}\right)^{-1} \end{aligned}$$

Exponential multi machine model

- Inter-arrival time is exponential with mean $t_a = 1/\lambda$
- Process time is exponential with mean $t_e = 1/\mu$
- m parallel identical machines

Then

$$\varphi = \frac{Q}{1-u} \cdot \frac{t_e}{m} + t_e$$

with

- $u = t_e/(m \cdot t_a)$ (machine utilization)
- $Q = \frac{(mu)^m}{m!} \left(\frac{(mu)^m}{m!} + (1-u) \sum_{i=0}^{m-1} \frac{(mu)^i}{i!} \right)^{-1}$

Multi machine model

- Inter-arrival time is general with mean t_a and standard deviation σ_a
- Process time is general with mean t_e and standard deviation σ_e
- m parallel identical machines

Then

$$\varphi \approx \gamma \cdot \frac{Q}{1-u} \cdot \frac{t_e}{m} + t_e$$

with

- $u = t_e/(m \cdot t_a)$ (machine utilization)
- $Q = \frac{(mu)^m}{m!} \left(\frac{(mu)^m}{m!} + (1-u) \sum_{i=0}^{m-1} \frac{(mu)^i}{i!} \right)^{-1}$
- $\gamma = \frac{1}{2} \cdot (c_a^2 + c_e^2)$
- $c_a = \sigma_a / t_a$
- $c_e = \sigma_e / t_e$

Multi machine model

Simple approximation for Q

$$Q \approx u^{\sqrt{2(m+1)}-1}$$

Multi machine model

- Inter-arrival time is general with mean t_a and standard deviation σ_a
- Process time is general with mean t_e and standard deviation σ_e
- m parallel identical machines

Then

$$\varphi \approx \gamma \cdot \frac{u^{\sqrt{2(m+1)}-1}}{1-u} \cdot \frac{t_e}{m} + t_e$$

with

- $u = t_e/(m \cdot t_a)$ (utilization per machine)
- $\gamma = \frac{1}{2} \cdot (c_a^2 + c_e^2)$
- $c_a = \sigma_a / t_a$
- $c_e = \sigma_e / t_e$

Departure process single machine

- Inter-arrival time is general with mean t_a and standard deviation σ_a
- Process time is general with mean t_e and standard deviation σ_e
- Single machine

Inter-departure time has mean t_d and standard deviation σ_d

- Output rate is input rate (conservation of flow)

$$t_d = t_a$$

- Coefficient of variation $c_d = \sigma_d / t_d$ is approximately equal to

$$c_d^2 \approx (1 - u^2) \cdot c_a^2 + u^2 \cdot c_e^2 \quad \text{(weighted average of } c_a^2 \text{ and } c_e^2\text{)}$$

with $c_a = \sigma_a / t_a$, $c_e = \sigma_e / t_e$ and $u = t_e / t_a$

Departure process multi machines

- Inter-arrival time is general with mean t_a and standard deviation σ_a
- Process time is general with mean t_e and standard deviation σ_e
- m parallel identical machines

Inter-departure time has mean t_d and standard deviation σ_d

- Output rate is input rate (conservation of flow)

$$t_d = t_a$$

- Coefficient of variation $c_d = \sigma_d / t_d$ is approximately equal to

$$c_d^2 \approx 1 + (1 - u^2)(c_a^2 - 1) + \frac{u^2}{\sqrt{m}}(c_e^2 - 1)$$

with $c_a = \sigma_a / t_a$, $c_e = \sigma_e / t_e$ and $u = t_e / (m \cdot t_a)$