Analysis of Manufacturing Systems 4AB00

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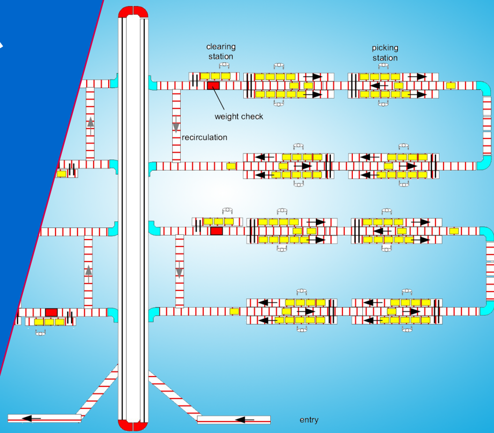

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Contents

Lecture notes on Analysis of Manufacturing Systems

- Chapter 3: 3.1, 3.2, 3.3

Recap: Process time variability

- Effective process time is total time seen by a job at a station
- Coefficient of variation c is relative measure of variability

  $$c = \sigma / t$$

  where t is the mean and σ the standard deviation
- Preemptive breakdowns:

  $$\begin{aligned} A &= \frac{m_f}{m_f + m_r} \quad \text{(availability)} \\ t_e &= \frac{t_0}{A} \\ c_e^2 &= c_0^2 + A(1-A)\frac{m_r}{t_0} + c_r^2 A(1-A)\frac{m_r}{t_0} \end{aligned}$$

  where
  - t_0 and c_0 are mean and cv of natural process time
  - m_f is mean exponential time to failure
  - m_r and σ_r are mean and standard deviation of time to repair

Recap: Process time variability

Example: What is better: short frequent stops or long rare ones?

Suppose $t_0 = \sigma_0 = 10$ minutes

Machine M_1: $m_f = 90$, $m_r = 10$, $\sigma_r = 0$

Machine M_2: $m_f = 900$, $m_r = 100$, $\sigma_r = 0$

Both machines have same availability $A = 0.9$, so same effective capacity

$$r_e = \frac{1}{t_e} = \frac{A}{t_0} = \frac{0.9}{10} = 0.09$$

but $c_e^2 = 1.09$ for M_1 and $c_e^2 = 1.9$ for M_2!

Flow variability

- Flow refers to transfer of jobs from one station to another
- t_a and σ_a are mean and standard deviation of time between arrivals
- Arrival rate

$$r_a = \frac{1}{t_a}$$

- Coefficient of variation of time between arrivals

$$c_a = \frac{\sigma_a}{t_a}$$

Low c_a arrivals

High c_a arrivals

Poisson flow

- Times between arrivals are independent and exponential with rate λ
- Memoryless property

$$P(\text{arrival in } (t, t+\Delta)) = 1 - e^{-\lambda\Delta} \approx \lambda\Delta$$

So in each small interval Δ is an arrival with probability $\lambda\Delta$!

- Dividing $(0, t)$ into many small intervals of length Δ, the number of arrivals in $(0, t)$ is binomial with $n = t/\Delta$ and $p = \lambda\Delta$
- Since n is large and p is small, this number is Poisson distributed with parameter $np = \lambda t$ (as Δ tends to 0)

$$P(k \text{ arrivals in } (0, t)) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \ldots$$

Poisson flow

- Since density $f(x) = \lambda e^{-\lambda x}$ is maximal for $x = 0$, short inter-arrival times occur more frequently than long ones. So arrivals tend to cluster:

- Superposition of many independent rarely occurring arrival flows is (close to) Poisson: this is why Poisson flows often occur in practice!
- Merging of two Poisson flows with rates λ_1 and λ_2 is again Poisson with rate $\lambda_1 + \lambda_2$, since

$$P(\text{arrival in } (t, t + \Delta)) \approx (\lambda_1 + \lambda_2)\Delta .$$

- Random splitting of Poisson flows with rate λ and splitting probability p is again Poisson with rate $p\lambda$, since

$$P(\text{arrival in } (t, t + \Delta)) \approx p\lambda\Delta .$$

Poisson flow

Example:

- Two type of jobs arrive at machine for processing, type A and B
- Both job types arrive according to Poisson flows
- Type A jobs arrive at rate 2 jobs per hour, type B with rate 3

Questions:

- What is the probability that during 1 hour no jobs arrive?
- What is the probability that the next job to arrive is type A?
- What is the probability that during 2 hours at least 2 type B jobs arrive?

Departure flow variability

- t_d is mean time between departures from workstation
- σ_d is standard deviation of time between departures from workstation
- Departure rate

$$r_d = \frac{1}{t_d}$$

- Coefficient of variation of time between departures

$$c_d = \frac{\sigma_d}{t_d}$$

- In serial production lines:

  Departures from workstation i are arrivals to workstation $i+1$, so

$$t_a(i+1) = t_d(i), \quad c_a(i+1) = c_d(i)$$

Departure flow variability

- Utilization of workstation with identical m machines

$$u = \frac{r_a t_e}{m}$$

- Single machine workstation ($m = 1$)

$$c_d^2 \approx (1 - u^2) c_a^2 + u^2 c_e^2 \quad \text{(weighted average of } c_a^2 \text{ and } c_e^2\text{)}$$

- Multi machine workstation ($m > 1$)

$$c_d^2 \approx 1 + (1 - u^2)(c_a^2 - 1) + \frac{u^2}{\sqrt{m}}(c_e^2 - 1)$$

Departure flow variability

Remarks:

- Times between departures are approximately independent
- If inter-arrival and process times are exponential ($c_a = c_e = 1$), then inter-departure times are independent and exponential ($c_d = 1$)

Question: Does the approximation for c_d make sense?

- If $u \approx 1$ and $m = 1$, then machine nearly always busy, so

$$c_d \approx c_e$$

- If $u \approx 0$, then t_e is very small compared to t_a, so

$$c_d \approx c_a$$

- If $c_a = c_e = 1$, then $c_d = 1$ (as it should)

Variability interactions - Queueing

- Building blocks for describing effects of variability in production lines:
  - Process time variability
  - Flow variability
- Question: How to evaluate impact of variability on key performance?
  - WIP
  - cycle time
  - throughput
- Note: Process time is often small part of cycle time, extra time is waiting
- Fundamental issue is to understand the underlying causes of waiting
- Science of waiting: Queueing theory
- Queueing system consists of three components:
  - arrival process
  - service (production) process
  - queue (buffer)

Single machine model

Components:

- Generator G sends jobs to machine M;
- Machine M processes these jobs and sends finished jobs to exit E;
- Exit E is doing some book keeping.

Parameters:

- Mean inter-arrival time t_a of the generator G;
- Mean process time t_e of the machine M.

Question: What is the throughput δ?

Note: This model can also be seen as two-machine zero-buffer mode

Object type job


```
type job = int;
```

Jobs are numbered.

Generator G


```
proc G(chan! job a; real ta):
    job x;

    while true:
        a!x;
        delay ta;
        x = x + 1;
    end
end
```

G generates jobs with constant inter-arrival times t_a.

Machine M


```
proc M(chan? job a; chan! job b; real te):
    job x;

    while true:
        a?x;
        delay te;
        b!x;
    end
end
```

M processes jobs with constant process times t_e.

Exit E


```
proc real E(chan? job a; int n):
    job x;

    while x < n:
        a?x;
    end;
    exit x / time
end
```

Exit E computes throughput over first n jobs.

Single machine model GME


```
model real GME(real ta, te; int n):
    chan job a, b;

    run G(a, ta), M(a, b, te), E(b, n)
end
```

Single machine model

Question: Why is δ for random process times smaller than for constant times?

Let A be inter-arrival time, B the (effective) process time and C cycle time,

$$C = \max\{A, B\}.$$

Then

$$\delta = \frac{1}{E(C)} \leq \frac{1}{\max\{E(A), E(B)\}}$$

with equality only for constant A and B.

Randomness leads to starvation ($A > B$) and blocking ($A < B$)!

Single machine model

Question: How to deal with variations in inter-arrival times and process times?

Answer: Use buffers!

If buffer is sufficiently large, then

$$\delta = \frac{1}{\max\{E(A), E(B)\}}.$$

Buffer machine model $GBME$


```
proc B(chan? job a; chan! job b; int N):
    list job xs;
    job x;
    while true:
        select
            size(xs) < N, a?x:
                xs = xs + [x]
        alt
            size(xs) > 0, b!xs[0]:
                xs = xs[1:]
        end
    end
end

model real GBME(real ta, te; int n, N):
    chan job a, b;
    run G(a, ta), B(a, b, N), M(b, c, te), E(c, n)
end
```

Question: How to calculate throughput in case of buffers?
Answer: Use queueing theory!

Exponential buffer machine model

- Inter-arrival time is exponential with rate λ
- Process time is exponential with rate μ
- Maximal n jobs in system (including one in G and M).

p_i is long-run probability (or fraction of time) of finding i jobs in the system

Question: How to determine these probabilities?

Answer: Through balance equations!

Exponential buffer machine model

State i means a total of i jobs in the system, 1 in process and $i-1$ waiting

Note: One of the waiting jobs in state n occupies machine G (which is blocked)

Balance equations

Flow from state i to $i+1$ = Flow from state $i+1$ to i

This yields

$$p_i \lambda = p_{i+1} \mu, \quad i = 0, \ldots, n-1,$$

Exponential buffer machine model

Solution

$$p_i = p_0 \left(\frac{\lambda}{\mu}\right)^i, \quad i = 0, \ldots, n$$

and p_0 follows from normalization

$$1 = \sum_{i=0}^{n} p_i = p_0 \frac{1 - (\lambda/\mu)^{n+1}}{1 - \lambda/\mu}$$

so

$$p_0 = \frac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{n+1}}.$$

The machine utilization is $u = 1 - p_0$ and the throughput

$$\delta = u\mu = (1 - p_0)\mu$$

Infinite capacity buffer model

Now let buffer capacity n tend to infinity

$$p_0 = 1 - \frac{\lambda}{\mu}$$

and for all $i = 0, 1, 2, \ldots$

$$p_i = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^i.$$

Number in system has geometric distribution with parameter λ/μ!

Remarks:

- We have to require

  $$\lambda < \mu$$

  otherwise the system will explode
- Interpretation: p_i is long-run fraction of time of finding i jobs in the system

Infinite capacity buffer model

- Throughput

$$\delta = \lambda$$

- Utilization

$$u = 1 - p_0 = \frac{\lambda}{\mu}$$

- Mean Work-In-Process (WIP) level

$$w = \sum_{i=0}^{\infty} i p_i = \frac{\lambda/\mu}{1 - \lambda/\mu} = \frac{u}{1-u}$$

- Mean Flow time

$$\varphi = \frac{w}{\lambda} = \frac{1/\mu}{1 - \lambda/\mu} = \frac{1/\mu}{1-u}$$

- Question: What happens to w and φ as $u \uparrow 1$?

Infinite capacity buffer model


```
proc B(chan? job a; chan! job b):
    list job xs;
    job x;
    while true:
        select
            a?x:
                xs = xs + [x]
        alt
            size(xs) > 0, b!xs[0]:
                xs = xs[1:]
        end
    end
end
model real GIBME(real ta, te; int n):
    chan job a, b;
    run G(a, exponential(ta)), B(a, b),
        M(b, c, exponential(te)), E(c, n)
end
```

Mean flow time

- Inter-arrival time is exponential with mean $t_a = 1/\lambda$
- Process time is exponential with mean $t_e = 1/\mu$
- Infinite capacity buffer
- Single machine
- Processing in order of arrival

Then

$$\varphi = \left(\frac{u}{1-u} + 1\right) \cdot t_e$$

with

- $u = t_e/t_a = \frac{\lambda}{\mu}$

Mean flow time

- Inter-arrival time is exponential with mean $t_a = 1/\lambda$
- Process time is general with mean t_e and standard deviation σ_e
- Infinite capacity buffer
- Single machine
- Processing in order of arrival

Then

$$\varphi = \left(\gamma \cdot \frac{u}{1-u} + 1 \right) \cdot t_e$$

with

- $u = t_e / t_a = \lambda t_e$
- $\gamma = \frac{1}{2} \cdot (1 + c_e^2)$
- $c_e = \sigma_e / t_e$
- Note that $c_e = 1$ for exponential process times!

Mean flow time

- Inter-arrival time is general with mean t_a and standard deviation σ_a
- Process time is general with mean t_e and standard deviation σ_e
- Infinite capacity buffer
- Single machine
- Processing in order of arrival

Then

$$\varphi \approx \left(\gamma \cdot \frac{u}{1-u} + 1\right) \cdot t_e$$

with

- $u = t_e / t_a$
- $\gamma = \frac{1}{2} \cdot (c_a^2 + c_e^2)$
- $c_a = \sigma_a / t_a$
- $c_e = \sigma_e / t_e$

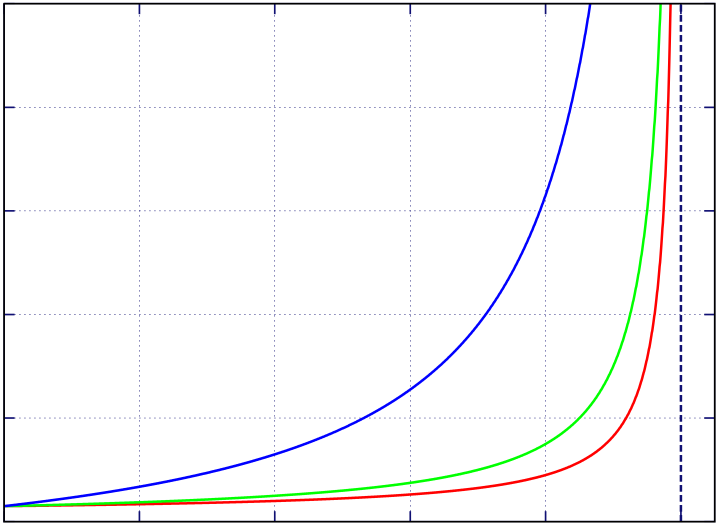
Mean flow time

$$\varphi \approx \left(\gamma \cdot \frac{u}{1-u} + 1\right) \cdot t_e \tag{1}$$

Lessons:

- As u tends to 1 then φ tends to ∞
- As u tends to 1, then approximation (1) becomes exact:
  relative error tends to 0;
- As u tends to 1, then *distribution* of flow time becomes exponential
- Summary:
  As system operates close to its maximum capacity,
  flow times are long and exponential with mean (1)
- Insensitivity: Mean flow time only depends on mean and standard deviation of inter-arrival times and process times!

Mean flow time



Mean flow time φ as function of utilization u, for $0 < u < 1$, and

$$c_a = 1, \quad c_s = 0, 1, 3.$$ Who is who?