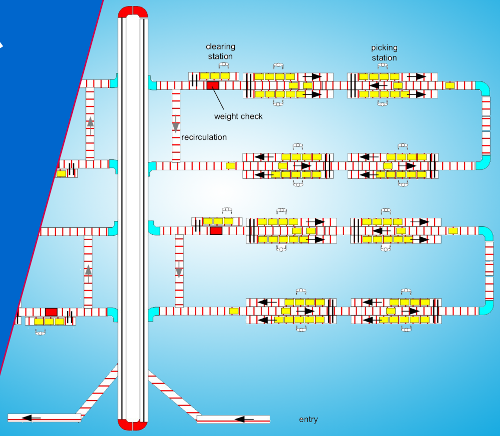
Analysis of Manufacturing Systems 4AB00

Ivo Adan



Technische Universiteit Eindhoven University of Technology

Contents

Lecture notes on Analysis of Manufacturing Systems

- Chapter 3: 3.3, 3.5
- Chapter 6
- Chapter 7

Recap: Conditional expectation

Conditional expectation of X given $Y = y$:

- Discrete random variables X and Y

$$E(X|Y = y) = \sum_x x P(X = x|Y = y),$$

  and

$$E(X) = \sum_y E(X|Y = y) P(Y = y).$$

- Continuous random variables X and Y

$$E(X|Y = y) = \int_{-\infty}^{\infty} x f_X(x|y) dx,$$

  and

$$E(X) = \int_{-\infty}^{\infty} E(X|Y = y) f_Y(y) dy.$$

Recap: Conditional expectation

Example:

A batch consists of n items with probability $(1-p)p^{n-1}, n \geq 1$.
The production time of a single item is uniform between 4 and 10 minutes.

- What is the mean production time of a batch?

Example: Process time variability

- Effective process time is total time seen by a job at a station
- Preemptive breakdowns:

$$
\begin{aligned}
A &= \frac{m_f}{m_f + m_r} \quad \text{(availability)} \\
t_e &= \frac{t_0}{A} \\
\sigma_e^2 &= \left(\frac{\sigma_0}{A}\right)^2 + \frac{(m_r^2 + \sigma_r^2)(1 - A)t_0}{A m_r}
\end{aligned}
$$

  where

  - t_0 and c_0 are mean and cv of natural process time
  - m_f is mean exponential time to failure
  - m_r and σ_r^2 are mean and variance of time to repair
- Question: How can we show that above formulas are correct?

Example: Random sums

$R_1, R_2, \ldots$ independent repair times with the same distribution

$$E(R) = m_r, \quad \mathrm{var}(R) = \sigma_r^2$$

- Fixed number n of repair times

$$E\left(\sum_{i=1}^{n} R_i\right) = n m_r, \quad \sigma^2\left(\sum_{i=1}^{n} R_i\right) = n\sigma_r^2$$

- Random number N of repair times

$$E\left(\sum_{i=1}^{N} R_i\right) = ?, \quad \sigma^2\left(\sum_{i=1}^{N} R_i\right) = ?$$

- Condition on $N = n$!

Example: Random sums

$R_1, R_2, \ldots$ independent repair times with the same distribution

$$E(R) = m_r, \quad \text{var}(R) = \sigma_r^2$$

- $P(N = n) = p_n, n = 1, 2, \ldots$

$$\begin{aligned} E\left(\sum_{i=1}^{N} R_i\right) &= \sum_{n=1}^{\infty} E\left(\sum_{i=1}^{N} R_i | N = n\right) P(N = n) \\ &= \sum_{n=1}^{\infty} E\left(\sum_{i=1}^{n} R_i\right) p_n \\ &= \sum_{n=1}^{\infty} n E(R) p_n \\ &= E(N) E(R) \\ &= E(N) m_r \end{aligned}$$

Example: Random sums

$R_1, R_2, \ldots$ independent repair times with the same distribution

$$E(R) = m_r, \quad \text{var}(R) = \sigma_r^2$$

- $P(N = n) = p_n, n = 1, 2, \ldots$

$$\begin{aligned}
E\left(\left(\sum_{i=1}^{N} R_i\right)^2\right) &= \sum_{n=1}^{\infty} E\left(\left(\sum_{i=1}^{N} R_i\right)^2 | N = n\right) P(N = n) \\
&= \sum_{n=1}^{\infty} E\left(\left(\sum_{i=1}^{n} R_i\right)^2\right) p_n \\
&= \sum_{n=1}^{\infty} (nE(R^2) + n(n-1)(E(R))^2) p_n \\
&= E(N)E(R^2) + (E(N^2) - E(N))(E(R))^2
\end{aligned}$$

Example: Random sums

$R_1, R_2, \ldots$ independent repair times with the same distribution

$$E(R) = m_r, \quad \text{var}(R) = \sigma_r^2$$

- $P(N = n) = p_n, n = 1, 2, \ldots$

$$\begin{aligned}
\text{var}\left(\sum_{i=1}^{N} R_i\right) &= E\left(\left(\sum_{i=1}^{N} R_i\right)^2\right) - \left(E\left(\sum_{i=1}^{N} R_i\right)\right)^2 \\
&= E(N)E(R^2) + (E(N^2) - E(N))(E(R))^2 - (E(N)E(R))^2 \\
&= E(N)\text{var}(R) + \text{var}(N)(E(R))^2 \\
&= E(N)\sigma_r^2 + \text{var}(N)m_r^2
\end{aligned}$$

Example: Process time variability

- Constant natural process time t_0 (so $c_0 = 0$)
- Time to failure is exponential with rate $1/m_f$
- Preemptive breakdowns: N is number of breakdowns during t_0
- N is Poisson(t_0/m_f)

$$
\begin{aligned}
E(N) &= \mathrm{var}(N) = \frac{t_0}{m_f} \\
t_e &= t_0 + \frac{t_0}{m_f} m_r \\
&= \frac{t_0}{A} \\
\sigma_e^2 &= \frac{t_0}{m_f}\sigma_r^2 + \frac{t_0}{m_f} m_r^2 \\
&= \frac{(m_r^2 + \sigma_r^2)(1 - A)t_0}{A m_r}
\end{aligned}
$$

Recap: Variability

- Variability is a fact of life
- There are many sources of variability in manufacturing systems
- Coefficient of variation is a key measure of variability
- Waiting time is often largest component of cycle time

Recap: Process time variability

- Effective process time is total time seen by a job at a station, includes
  - natural process time
  - setups
  - rework
  - operator unavailability
  - breakdowns
  - and other shop floor realities
- Standard deviation σ is absolute measure of variability
- Coefficient of variation c is relative measure of variability

$$c = \frac{\sigma}{t}$$

where t is the mean and σ the standard deviation

Recap: Flow variability

- Flow refers to transfer of jobs from one station to another
- t_a and σ_a are mean and standard deviation of time between arrivals
- Arrival rate

$$r_a = \frac{1}{t_a}$$

- Coefficient of variation of time between arrivals

$$c_a = \frac{\sigma_a}{t_a}$$

Low c_a arrivals

High c_a arrivals

Recap: Poisson flow

- Times between arrivals are independent and exponential with rate λ
- Memoryless property

$$P(\text{arrival in } (t, t+\Delta)) = 1 - e^{-\lambda\Delta} \approx \lambda\Delta$$

So in each small interval Δ is an arrival with probability $\lambda\Delta$!

- Dividing $(0, t)$ into many small intervals of length Δ, the number of arrivals in $(0, t)$ is binomial with $n = t/\Delta$ and $p = \lambda\Delta$
- Since n is large and p is small, this number is Poisson distributed with parameter $np = \lambda t$ (as Δ tends to 0)

$$P(k \text{ arrivals in } (0, t)) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \ldots$$

Recap: Variability interactions

- Inter-arrival time is general with mean t_a and standard deviation σ_a
- Process time is general with mean t_e and standard deviation σ_e
- m parallel identical machines

Then

$$\varphi \approx \gamma \cdot \frac{u^{\sqrt{2(m+1)}-1}}{1-u} \cdot \frac{t_e}{m} + t_e$$

with

- $u = t_e/(m \cdot t_a)$ (utilization per machine)
- $\gamma = \frac{1}{2} \cdot (c_a^2 + c_e^2)$
- $c_a = \sigma_a / t_a$
- $c_e = \sigma_e / t_e$

Recap: Variability

- Variability propagates

Recap: Departure process

- Inter-arrival time is general with mean t_a and standard deviation σ_a
- Process time is general with mean t_e and standard deviation σ_e
- m parallel identical machines

Inter-departure time has mean t_d and standard deviation σ_d

- Output rate is input rate (conservation of flow)

$$t_d = t_a$$

- Coefficient of variation $c_d = \sigma_d / t_d$ is approximately equal to

$$c_d^2 \approx 1 + (1 - u^2)(c_a^2 - 1) + \frac{u^2}{\sqrt{m}}(c_e^2 - 1)$$

with $c_a = \sigma_a / t_a$, $c_e = \sigma_e / t_e$ and $u = t_e / (m \cdot t_a)$

Splitting and merging

- Random splitting of flow: If a flow with rate λ and cv c is randomly split (or finned) with probability p, then resulting flow has rate $p\lambda$ and cv c_a

$$c_a^2 = pc^2 + 1 - p$$

- Merging flows: If two flows with rates λ_1 and λ_2 and cv c_1 and c_2 are merged, then resulting flow has rate $\lambda_1 + \lambda_2$ and cv c_a

$$c_a^2 \approx \frac{\lambda_1}{\lambda_1 + \lambda_2}\, c_1^2 + \frac{\lambda_2}{\lambda_1 + \lambda_2}\, c_2^2$$

Note:

Poisson flows remain Poisson after merging or random splitting

Blocking: Buffer machine model

- Process time machine 1 is exponential with rate λ
- Process time machine 2 is exponential with rate μ
- Buffer of $b-2$ jobs in between two machines
- Machine 1 never starved (always raw material)
- Machine 2 never blocked

Analysis:

- Two machine system can be in states $i=0,1,\dots,b$
- State i means total of i jobs in the system,
  with 1 job in process by machine 1 and $i-1$ waiting in the buffer
- In state b machine 1 is blocked and occupied by waiting job

Blocking: Buffer machine model

- Process time machine 1 is exponential with rate λ
- Process time machine 2 is exponential with rate μ
- Buffer of $b-2$ jobs in between two machines
- Machine 1 never starved (always raw material)
- Machine 2 never blocked

Analysis:

- If $u = \lambda/\mu \neq 1$

$$p_i = u^i \frac{1-u}{1-u^{b+1}}, \quad i = 0, 1, \ldots, b$$

- if $u = 1$

$$p_i = \frac{1}{b+1}, \quad i = 0, 1, \ldots, b$$

Blocking: Buffer machine model

- Process time machine 1 is exponential with rate λ
- Process time machine 2 is exponential with rate μ
- Buffer of $b-2$ jobs in between two machines
- Machine 1 never starved (always raw material)
- Machine 2 never blocked

Analysis:

- Mean Work-In-Process (WIP) level

$$w = \sum_{i=0}^{b} i p_i = \frac{u}{1-u} - \frac{(b+1)u^{b+1}}{1-u^{b+1}} \quad (u \neq 1)$$

- Throughput

$$\delta = \frac{1-u^b}{1-u^{b+1}}\lambda = \frac{1-u^b}{1-u^{b+1}}u\mu \quad (u \neq 1)$$

Blocking: Buffer machine model

- Process time machine 1 is exponential with rate λ
- Process time machine 2 is exponential with rate μ
- Buffer of $b-2$ jobs in between two machines
- Machine 1 never starved (always raw material)
- Machine 2 never blocked

Insights:

- Finite buffers force stability, regardless λ and μ
- WIP is always less than in infinite buffer system

$$w < \frac{u}{1-u}$$

- Throughput is always less than in infinite buffer system

$$\delta < u\mu$$

Blocking: Buffer machine model

- Process time machine 1 is exponential with rate λ
- Process time machine 2 is exponential with rate μ
- Buffer of $b-2$ jobs in between two machines
- Machine 1 never starved (always raw material)
- Machine 2 never blocked

Insights:

- Only way to reduce WIP without sacrificing too much throughput is:

  variability reduction

Blocking: Buffer machine model

Example

- $1/\lambda = 21$ minutes, $1/\mu = 20$ minutes
- $u = 20/21 = 0.9524$
- $b = \infty$:

  $w = 20$ jobs, $\delta = 0.0476$ jobs/minute, $\varphi = 420.14$ minutes

- $b = 4$:

  $w = 1.894$ jobs, $\delta = 0.039$ jobs/minute, $\varphi = 48.57$ minutes

- $1/\lambda = 20$ minutes, $1/\mu = 21$ minutes
- $u = 21/20 = 1.05$
- $b = 4$:

  $w = 2.097$ jobs, $\delta = 0.039$ jobs/minute, $\varphi = 53.78$ minutes