Analysis of Manufacturing Systems 4AB00

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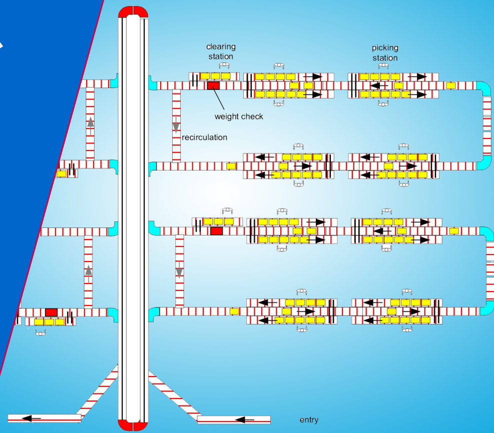

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- Chapter 5: 5.1, 5.2 (till 5.2.3)
- Chapter 10: 10.1, 10.2, 10.3, 10.4 (till 10.4.8), 10.5, 10.6

Recap

- Ingredients of a probability model
  - Sample space S, which can be discrete or continuous
  - Events, which are subsets of S
  - Probabilities $P(E)$ of events
- Conditional probability

$$P(E|F) = \frac{P(EF)}{P(F)}, \text{ or } P(EF) = P(E|F)P(F)$$

- Independent events E and F

$$P(EF) = P(E)P(F)$$

Recap

- Discrete random variable X takes discrete values $x_1, x_2, \ldots$
- Function $p_j = P(X = x_j)$ is probability mass function
- Continuous random variable X has density $f(x)$ and distribution function

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)dy$$

- Interpretation of density: $P(x < X \leq x + dx) \approx f(x)dx$
- Random variables X and Y are independent if for all x and y,

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

Recap

- Expected value (discrete)

$$E(X) = \sum_{j=1}^{\infty} x_j p_j$$

- Expected value (continuous)

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

- Expectation of sum = sum of expectation

$$E(X+Y) = E(X) + E(Y)$$

- Variance (measure of variability)

$$\begin{aligned} \text{var}(X) &= E\left((X - E(X))^2\right) \\ &= E(X^2) - (E(X))^2 . \end{aligned}$$

Boarding pass problem

100 people line up to board an airplane with 100 seats. Each passenger gets on one at a time to select his assigned seat. The first one has lost his boarding pass and takes a random seat. If subsequent passengers come and claim his seat, he apologizes and takes another random unoccupied seat.

You are the last passenger...

What is the probability that your seat is free?

Reliability

Consider a system composed of n components, and q_i is the probability that component i works (which is independent of the other components).

Serial system: All components have to work for the system to work.

Then the probability Q that the system works is

$$Q = q_1 q_2 \cdots q_n = \prod_{i=1}^{n} q_i .$$

Parallel (or redundant) system: At least one component has to work for the system to work.

Then the probability Q that the system works is

$$Q = 1 - \prod_{i=1}^{n} (1 - q_i).$$

Reliability of combined system

Serial system of length n, where part i is a parallel system of k_i components. Component (i, j) works with probability $q_{i,j}$. Then

$$Q_i = 1 - \prod_{j=1}^{k_i} (1 - q_{i,j})$$

is the probability that part i works and the probability that system works is

$$Q = \prod_{i=1}^{n} Q_i .$$

Example: Improve reliability of serial system.

Which option to prefer?

- K copies of the serial system in parallel.
- K copies of each component in parallel.

Minimal redundancy

Serial system of length n, where part i is a parallel system of k_i identical components. Component (i, j) works with probability q_i. Then

$$Q_i(k_i) = 1 - (1 - q_i)^{k_i}$$

is the probability that part i works and the probability that system works is

$$Q(k_1, k_2, \ldots, k_n) = \prod_{i=1}^{n} Q_i(k_i).$$

Question: What are the minimal numbers k_i such that $Q(k_1, k_2, \ldots, k_n) > Q$?

Greedy (optimal) approach:

Start with $k_1 = \cdots = k_n = 1$ and then subsequently add a component for which $Q_i(k_i + 1)/Q_i(k_i)$ is maximal, until $Q(k_1, k_2, \ldots, k_n) > Q$.

Minimal redundancy

Example:

Serial system with 3 parts, $q_1 = 0.5$, $q_2 = 0.7$, $q_3 = 0.8$.
Target Q is 0.8.
What is the minimal number of components required?

i	1	2	3
$Q_i(1)$	0.500	0.700	0.800
$Q_i(2)$	0.750	0.910	0.960
$Q_i(3)$	0.875	0.973	0.992
$Q_i(4)$	0.938	0.992	0.998

i	1	2	3
$Q_i(2)/Q_i(1)$	1.50	1.30	1.20
$Q_i(3)/Q_i(2)$	1.17	1.06	1.03
$Q_i(4)/Q_i(3)$	1.07	1.02	1.01

You will need 5 extra components. Which?

Recap

- Uniform random variable X on (a, b),

$$f(x) = \frac{1}{b-a}, \quad a < x < b, \quad E(X) = \frac{1}{2}(a+b), \quad \mathrm{var}(X) = \frac{1}{12}(b-a)^2.$$

- Exponential random variable X with parameter (or rate) $\lambda > 0$,

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0, \quad E(X) = \frac{1}{\lambda}, \quad \mathrm{var}(X) = \frac{1}{\lambda^2}.$$

- Memoryless property for exponential X: for all $t, s > 0$,

$$P(X > t + s | X > s) = P(X > t).$$

- For independent exponentials $X_1, \ldots, X_n$ with rates $\lambda_1, \ldots, \lambda_n$,

$$P(\min_i X_i > t) = e^{-(\lambda_1 + \cdots + \lambda_n)t}, \quad t > 0.$$

So $\min\{X_1, \ldots, X_n\}$ is exponential with rate $\lambda_1 + \cdots + \lambda_n$.

Recap

Example:

One-time business decision: How much stock to order in order to meet random demand during a single period? The demand is a continuous random variable X with density $f(x) = \mu e^{-\mu x}$ for $x > 0$.

Questions:

- Suppose you order Q units.
  What is the (stockout) probability that the stock Q will not be enough to meet demand?
- How to choose Q so that the stockout probability is no more than 10%?

Recap

Example:

The time (in hundreds of hours) until failure of power supply to a radar system is a random variable X with probability density function

$$f(x) = \begin{cases} \frac{1}{625}(x - 50) & \text{for } 50 < x \leq 75, \\ \frac{1}{625}(100 - x) & \text{for } 75 < x \leq 100, \end{cases}$$

and $f(x) = 0$ otherwise (triangular distribution).

Question:

What is the expected value of X?

Sums of exponentials

Let $X_1, \ldots, X_n$ be independent exponentials with rates $\lambda_1 = \cdots = \lambda_n = \lambda$.

- The density of the sum $X = X_1 + \cdots + X_n$ is

$$f(x) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \quad t > 0,$$

and

$$P(X \leq t) = 1 - \sum_{i=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^i}{(i)!}.$$

This is the Erlang distribution.

Continuous random variables

- Gamma random variable X with parameters $\alpha > 0$ and $\lambda > 0$,

$$f(x) = \frac{1}{\Gamma(\alpha)} \lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}, \quad x > 0,$$

and $f(x) = 0$ otherwise, where $\Gamma(a)$ is the *gamma function*,

$$\Gamma(a) = \int_0^{\infty} e^{-y} y^{a-1} dy, \quad a > 0.$$

Then

$$E(X) = \frac{\alpha}{\lambda}, \quad \text{var}(X) = \frac{\alpha}{\lambda^2}.$$

Note: The gamma function has the property (integrate by parts)

$$\Gamma(a) = (a-1)\Gamma(a-1),$$

so, if $a = n$, then $\Gamma(n) = (n-1)!$, so Gamma(λ, n) = Erlang(λ, n)

Continuous random variables

- Normal random variable X with parameters μ and $\sigma > 0$,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}, \quad -\infty < x < \infty$$

Then

$$E(X) = \mu, \quad \text{var}(X) = \sigma^2.$$

Density $f(x)$ is denoted as $N(\mu, \sigma^2)$ density.

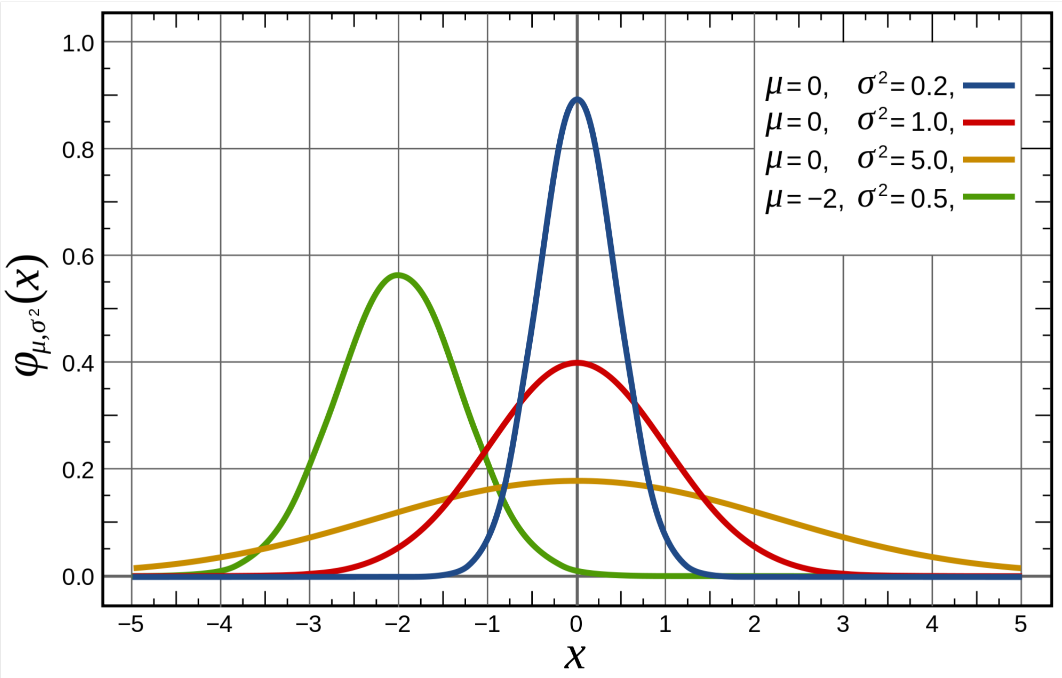
- Standard normal random variable X has $N(0, 1)$ density, so

$$f(x) = \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

and

$$P(X \le x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy.$$

Normal distribution



Properties of normals

- Linearity: If X is normal, then $aX + b$ is normal.
- Additivity: If X and Y are independent and normal, then $X + Y$ is normal.

Question: What are the parameters of $aX + b$ and $X + Y$?

- Probability that X lies $\geq z$ standard deviations above its mean is

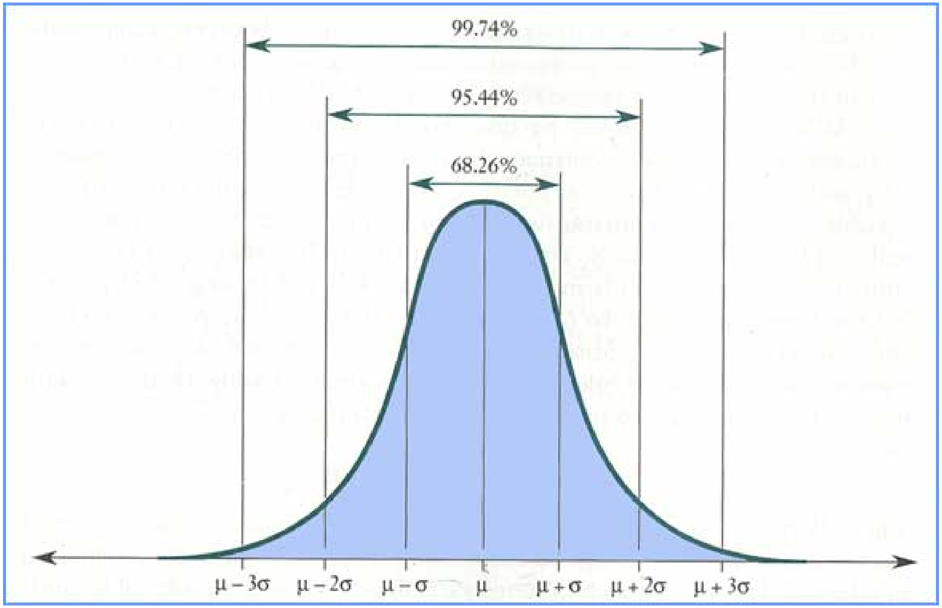
$$P(X \geq \mu + z\sigma) = 1 - \Phi(z).$$

- $100p\%$ percentile z_p of standard normal distribution is solution of

$$\Phi(z_p) = p.$$

For example, $z_{0.95} = 1.64$, $z_{0.975} = 1.96$.

Properties of normals



Normal (or Gaussian) distribution

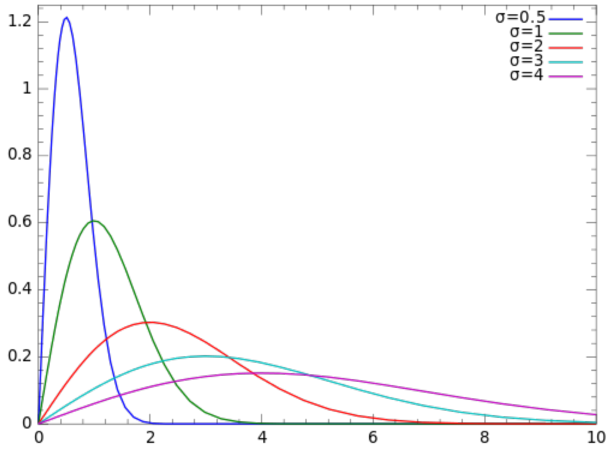
Transformations

Let X have density $f(x)$, and $v(x)$ be strictly increasing, with inverse $w(y)$.

Then the density of the random variable $Y = v(X)$ is

$$f(w(y))\frac{d}{dy}w(y).$$

Example: Let X be exponential with rate λ, then the density of $Y = \sqrt{X}$ is $f(y) = 2\lambda y e^{-\lambda y^2}$. This density is called the Rayleigh density.



$$\sigma^2 = 1/(2\lambda)$$

Inverse-transformation method

Let U be uniform on $(0, 1)$ and X have continuous increasing distribution F, with inverse F^{-1}.

Then $Y = F^{-1}(U)$ has the same distribution as X.

Simulating a random observation X:

- Generate U from $U(0, 1)$;
- Return $X = F^{-1}(U)$.

Example:

$X = -\frac{1}{\lambda}\log(U)$ is exponential with rate λ.

Failure rate

X with density $f(x)$ is the life time of an item.

Probability that item of age x will fail in the next Δ time units is

$$P(X \leq x + \Delta | X > x) = \frac{P(x < X \leq x + \Delta)}{P(X > x)} \approx \frac{f(x)\Delta}{1 - F(x)}.$$

The failure rate or hazard rate of X is defined as

$$r(x) = \frac{f(x)}{1 - F(x)},$$

which is the intensity that an item of age x will fail in the next moment.

Example: If X is exponential with rate λ, then $r(x) = \lambda$.

Example: If X is uniform on (a, b), then what is $r(x)$?

Many complex systems have bathtub-shaped failure rate.