Analysis of Manufacturing Systems 4AB00

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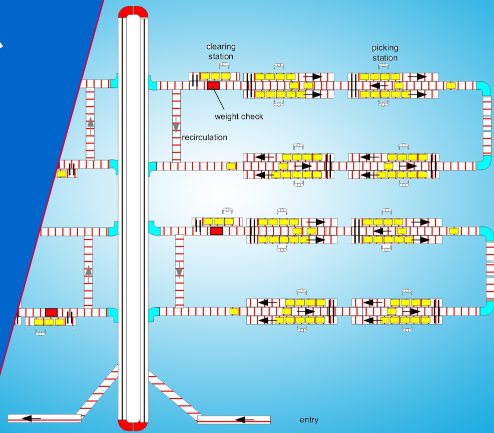

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Contents

- Chapter 10: 10.1, 10.2, 10.3, 10.4 (till 10.4.4)

Recap

- Ingredients of a probability model
  - Sample space S, which can be discrete or continuous
  - Events, which are subsets of S
  - Probabilities $P(E)$ of events
- Conditional probability

$$P(E|F) = \frac{P(EF)}{P(F)}, \text{ or } P(EF) = P(E|F)P(F)$$

- Independent events E and F

$$P(EF) = P(E)P(F)$$

Recap

- Discrete random variable X takes discrete values $x_1, x_2, \ldots$
- Function $p_j = P(X = x_j)$ is probability mass function
- Expected value (weighted average of outcomes)

$$E(X) = \sum_{j=1}^{\infty} x_j p_j$$

- Variance (measure of variability)

$$\begin{aligned} \text{var}(X) &= E\left((X - E(X))^2\right) \\ &= E(X^2) - (E(X))^2 . \end{aligned}$$

Recap

- Expectation of sum = sum of expectation

$$E(X+Y) = E(X) + E(Y)$$

- Random variables X and Y are independent if for all x_i, y_j,

$$P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j)$$

- Variance of sum = sum of variance (only if X and Y are independent!)

$$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y).$$

Recap

- Bernoulli random variable X with success probability p,

$$P(X = 1) = 1 - P(X = 0) = p.$$

- Binomial random variable X of n independent trials with success prob p,

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \ldots n.$$

- Poisson random variable X with parameter $\lambda > 0$,

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

- Geometric random variable X with success probability p,

$$P(X = k) = (1 - p)^{k-1} p, \quad k = 1, 2, \ldots$$

Recap

Example: KIVA system

- n positions (equally spaced) on one side of an aisle in the storage area
- Each position has width of 1 meter
- Each position holds one rack, except for one location which is open
- Robot Betty  always combines a storage and retrieval action
- When Betty enters aisle:
  - (open) storage location S is random
  - retieval location R is random

Questions:

- What is the probability that Betty has to travel up to k meter in the aisle? So what is $P(\max\{S, R\} = k)$?
- What is the expected distance Betty has to travel in the aisle?

Recap

Example: Batch quality check

- Production batch consists of n parts
- After production, each part is checked for quality
- Part successfully passes the test with probability q

Questions:

- What is the probability that k parts pass the test?
- What is the expected number of parts that pass the test?

Recap

Example: Production orders

- Production orders for two part types, a and b, arrive at workstation W
- Number of production orders per day for type a parts is Poisson(λ_a)
- For type b parts it is Poisson(λ_b)

Questions:

- What is the probability that k production orders (for a or b) arrive during the day?
- What is the expected number of production orders that arrive during the day?

Continuous random variable

Example:
Break stick of length 1 at random in two pieces and let X be the ratio of shorter piece and longer piece.

Then

$$F(x) = P(X \leq x) = \frac{2x}{1+x} = \int_{-\infty}^{x} f(y)dy, \quad 0 \leq x \leq 1,$$

where $f(x)$ is the derivative of $F(x)$,

$$f(x) = \frac{d}{dx}F(x) = \begin{cases} \frac{2}{(1+x)^2} & 0 \leq x \leq 1, \\ 0 & \text{else.} \end{cases}$$

$F(x)$ is probability distribution of X and $f(x)$ is probability density of X.

Continuous random variable

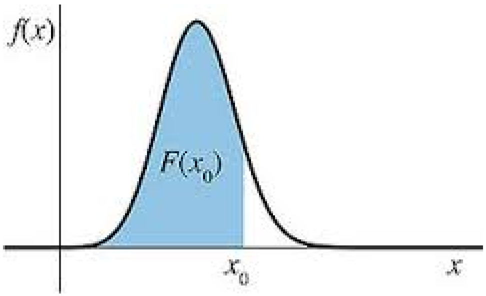
A random variable X is continuously distributed if

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)dy,$$

where the function $f(x)$ satisfies

$$f(x) \geq 0 \text{ for all } x, \qquad \int_{-\infty}^{\infty} f(x)dx = 1.$$

$F(x)$ is probability distribution of X and $f(x)$ is probability density of X.



Continuous random variable

For all $a < b$,

$$\begin{aligned} P(a < X \leq b) &= P(X \leq b) - P(X \leq a) \\ &= \int_{-\infty}^{b} f(x)dx - \int_{-\infty}^{a} f(x)dx \\ &= \int_{a}^{b} f(x)dx \end{aligned}$$

or more generally, for every set B,

$$P(X \in B) = \int_B f(y)dy,$$

Continuous random variable

- $P(X = b) = 0$ for all b, since

$$P(X = b) = \lim_{a \uparrow b} P(a < X \leq b) = \lim_{a \uparrow b} \int_a^b f(x)dx = \int_b^b f(x)dx = 0$$

- Interpretation of density: for small $\Delta > 0$

$$P(x < X \leq x + \Delta) = \int_x^{x+\Delta} f(y)dy \approx f(x)\Delta$$

Continuous random variable

A random variable X is continuously distributed if

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)dy,$$

Example:

- Uniform random variable X on $[0, 1]$,

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1, \\ 0 & \text{else.} \end{cases}$$

- Uniform random variable X on $[a, b]$,

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b, \\ 0 & \text{else.} \end{cases}$$

Finding densities

Recipe:

First determine $F(x) = P(X \leq x)$ and then differentiate.

Example:

- Let $X = -\ln(U)/\lambda$ where U is random on $(0, 1)$.
  What is density of X?
- X is distance to 0 of a random point in a disk of radius r.
  What is density of X?

Expected value

For a continuous random variable X with density $f(x)$,

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

is its expected value or expectation or first moment (assuming that it exists).

Example:

- X is distance to 0 of a random point in a disk of radius r.
  Then $E(X) = \frac{2}{3}r$.
- Break stick of length 1 at random in two pieces and let X be the ratio of shorter piece and longer piece. Then

$$E(X) = 2\ln(2) - 1.$$

Substitution rule

For any function g of the continuous X with density $f(x)$,

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

(assuming that it exists).

Example:

- Break stick of length 1 at random point U in two pieces and let X be the ratio of shorter piece and longer piece. Then

  $$E(X) = E(g(U)),$$

  where

  $$g(u) = \begin{cases} u/(1-u) & \text{for } 0 < u \leq \frac{1}{2}, \\ (1-u)/u & \text{for } \frac{1}{2} < u < 1. \end{cases}$$

Variance

The variance of X is

$$\text{var}(X) = E((X - E(X))^2) = \int_{-\infty}^{\infty} (x - E(X))^2 f(x)dx.$$

It is a measure of the spread of the possible values of X.

The standard deviation of X is the square root of the variance,

$$\sigma(X) = \sqrt{\text{var}(X)}.$$

It has the same units as $E(X)$.

Example:

- X is distance to 0 of a random point in a disk of radius r.
  Then $\text{var}(X) = \frac{1}{18}r^2$.
- Let X be random number in (a, b).
  What are $E(X)$ and $\text{var}(X)$?

Continuous random variables

- **Uniform** random variable X on (a, b),

$$f(x) = \frac{1}{b-a}, \quad a < x < b,$$

and $f(x) = 0$ otherwise. Then

$$P(X \leq t) = \frac{t-a}{b-a}, t > 0, \quad E(X) = \frac{1}{2}(a+b), \quad \mathrm{var}(X) = \frac{1}{12}(b-a)^2.$$

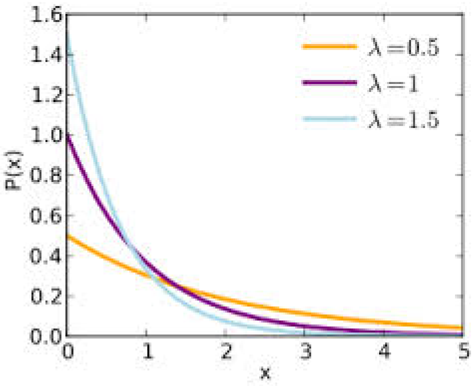
Continuous random variables

- Exponential random variable X with parameter (or rate) $\lambda > 0$,

  $$f(x) = \lambda e^{-\lambda x}, \quad x > 0,$$

  and $f(x) = 0$ otherwise. Then

  $$P(X \leq t) = 1 - e^{-\lambda t}, t > 0, \quad E(X) = \frac{1}{\lambda}, \quad \text{var}(X) = \frac{1}{\lambda^2}.$$



Properties of exponentials

- Memoryless property: for all $t, s > 0$,

$$P(X > t + s | X > s) = P(X > t).$$

This follows from

$$\begin{aligned} P(X > t + s | X > s) &= \frac{P(X > t + s, X > s)}{P(X > s)} \\ &= \frac{P(X > t + s)}{P(X > s)} \\ &= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} \\ &= e^{-\lambda t}. \end{aligned}$$

In words: Used is as good as new!

Reliability

Consider system of n components where life time of component i is exponential with parameter λ_i. Let $Q(t)$ denote probability that system works at time t.

Serial system: All components have to work for the system to work.

$$Q(t) = \prod_{i=1}^{n} e^{-\lambda_i t} = e^{-(\lambda_1 + \cdots + \lambda_n)t}.$$

Parallel system: At least one component has to work for the system to work.

$$Q(t) = 1 - \prod_{i=1}^{n} (1 - e^{-\lambda_i t}).$$

Properties of exponentials

Let $X_1, \ldots, X_n$ be independent exponentials with rates $\lambda_1, \ldots, \lambda_n$.

- $\min\{X_1, \ldots, X_n\}$ is exponential with rate $\lambda_1 + \cdots + \lambda_n$,

$$P(\min_i X_i > t) = e^{-(\lambda_1 + \cdots + \lambda_n)t}, \quad t > 0.$$

Properties of exponentials

Let $X_1, \ldots, X_n$ be independent exponentials with rates $\lambda_1, \ldots, \lambda_n$.

- Probability that $X_i = \min\{X_1, \ldots, X_n\}$ is proportional to λ_i,

$$P(X_i = \min_j X_j) = P(X_i < \min_{j \neq i} X_j) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}.$$

Properties of exponentials

Let $X_1, \ldots, X_n$ be independent exponentials with rates $\lambda_1, \ldots, \lambda_n$.

- Ordering of X_i and $\min_j X_j$ are independent,

$$
\begin{aligned}
P(X_i < \min_{j \neq i} X_j | \min_j X_j > t) &= P(X_i - t < \min_{j \neq i} X_j - t | \min_j X_j > t) \\
&= P(X_i < \min_{j \neq i} X_j).
\end{aligned}
$$

Size of minimum does not tell anything about whom of X_i is the minimum!