Analysis of Manufacturing Systems 4AB00

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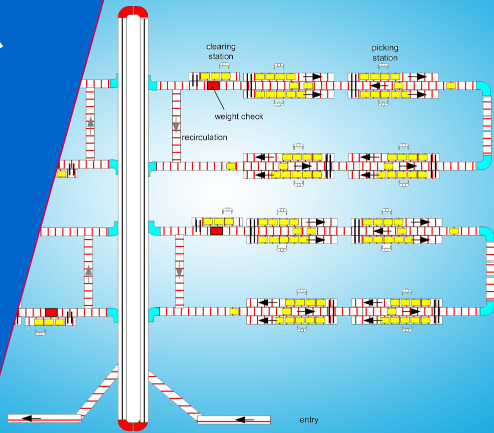

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Conditional probabilities

For any two event E and F with $P(F > 0)$,

$$P(E|F) = \frac{P(E \cap F)}{P(F)},$$

where we usually write $P(E \cap F) = P(EF)$.

Some intuition: Do an experiment n times.

Suppose F occurs r times with E, and s times without E.
Then the relative frequencies are

$$f_n(EF) = \frac{r}{n}, \quad f_n(F) = \frac{r+s}{n}$$

and the frequency $f_n(E|F)$ of E in those experiments in which F occurred,

$$f_n(E|F) = \frac{r}{r+s} = \frac{f_n(EF)}{f_n(F)}.$$

Conditional probabilities

Example: Rolling a die twice, $P(\{i, j\}) = \frac{1}{36}$.

- Given that $i = 4$ (event F), what is probability that $j = 2$ (event E)?

$$P(E|F) = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$$

- Given that one of the dice turned up with 6, what is the probability that the other one turned up with 6 as well?

Example: Darts on unit disk.

$$P(\text{distance to } 0 > \tfrac{1}{2} | x > 0) = \frac{\frac{1}{2}\pi - \frac{1}{8}\pi}{\frac{1}{2}\pi} = \frac{3}{4}$$

Conditional probabilities

Rewrite the formula for conditional probability in intuitive form:

$$P(EF) = P(E|F)P(F)$$

This is product rule and can be used to assign probabilities.

Example: Champions League.

Eight soccer teams reached quarter finales, two teams from each of the countries England, Germany, Italy and Spain. The matches are determined by drawing lots. What is the probability that the two teams from the same country play against each other in all four matches?

Repeated application of product rule:

$$P(E_1E_2\cdots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\cdots P(E_n|E_1E_2\cdots E_{n-1})$$

Independent events

Special case:

$$P(E|F) = P(E)$$

Then learning that F occurred does not change the probability of E.

Two events E and F are independent if

$$P(EF) = P(E)P(F)$$

Example: Coin tossing.

Two fair coins are tossed. E is event that first coin is H, F is event that both coin display same outcome. Are E and F independent?

Independent events

Example: Rolling a die twice

Take $E =$ "$i + j = 6$" and $F =$ "$i = 4$". Are E and F Independent?
Now take $E =$ "$i + j = 7$". Independent?

Independent experiments:

The sample space is

$$S = S_1 \times S_2 \times \cdots \times S_n = \{(s_1, \ldots, s_n),\ s_1 \in S_1, \ldots, s_n \in S_n\}$$

where

$$P(E_1 E_2 \ldots E_n) = P(E_1)P(E_2) \cdots P(E_n)$$

for events $E_1 \subset S_1, \ldots, E_n \subset S_n$.

Law conditional probability

Example: Random tossing a coin.

A die is rolled and then a fair coin is tossed that many times. What is the probability that Heads will not appear?

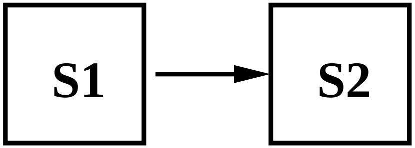
Let E be an event that can only occur if one of the mutually disjoint events $F_1, \ldots, F_n$ occurs. Then

$$P(E) = P(E|F_1)P(F_1) + \cdots + P(E|F_n)P(F_n).$$

Example: Tour de France.

The Tour de France will take place from July 1 to July 23. 180 cyclists participate. What is the probability that at least two cyclists will have birthdays on the same day during the tournament?

Tight two-station line



- Station 1 has 1 machine, daily production of this machine is $1, \ldots, 6$ units with equal probability
- Station 2 has 1 machine, daily production of this machine is 3 or 4 units with equal probabability

Questions:

- What is capacity of each station? Balanced line?
- What is expected daily throughput of the line?
- Add an identical machine to station 1. What is throughput?
- Add an identical machine to station 2. What is throughput?

Random variable

Function that assigns a numerical value to each outcome of an experiment: X

Examples:

- Rolling a die twice, X is sum of outcomes, so $X = i + j$
- Rolling a die twice, Y is maximum outcome, so $Y = \max(i, j)$
- Repeatedly flipping a coin, N is number of flips until first H

Discrete random variable X can only take (possibly infinitely many) discrete values, $x_1, x_2, \ldots$, and the function $p_j = P(X = x_j)$ is the probability mass function or probability distribution of X.

Examples:

- Rolling a die twice, $P(X = 2) = \frac{1}{36}$, $P(X = 3) = \frac{2}{36}$, $P(X = 5) = \frac{4}{36}$
- Rolling a die twice, $P(Y = 1) = \frac{1}{36}$, $P(Y = 2) = \frac{3}{36}$, $P(Y = 3) = \frac{5}{36}$
- Number of flips until first H, with $P(H) = 1 - P(T) = p$,

$$P(N = n) = (1 - p)^{n-1} p, \quad n = 1, 2, \ldots$$

Expected value

For a random variable X with probability mass function $p_j = P(X = x_j)$,

$$E(X) = \sum_{j=1}^{\infty} x_j p_j$$

is its expected value or expectation or first moment (assuming that it exists).

Example:

- Rolling a die, X is the number of points,

$$E(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} = 3.5$$

- Number of flips until first H, with $P(H) = 1 - P(T) = p$,

$$E(N) = \sum_{n=1}^{\infty} n(1-p)^{n-1} p = \frac{1}{p}$$

Expected value of sums

Property: For any two random variables X and Y

$$E(X+Y) = E(X) + E(Y)$$

and thus for any (finite) number of random variables $X_1, X_2, \ldots, X_n$,

$$E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n).$$

So: Expectation of sum = sum of expectation (always)!

Example:

n children of different lengths are placed in line at random. Start with first child and walk till end of line. When you encounter a taller child than seen so far, she will join you. Let X be the number that joined you. What is $E(X)$?

Substitution rule

Property: For any function g of X,

$$E(g(X)) = \sum_{x_i} g(x_i) P(X = x_i) = \sum_{x_i} g(x_i) p_i$$

Remark: In general, $E(g(X)) \neq g(E(X))$!

However (take $g(x) = ax + b$)

Linearity: For any constants a and b,

$$E(aX + b) = aE(X) + b$$

Variance

The expected value of $(X - E(X))^2$ is the variance of X,

$$\text{var}(X) = E((X - E(X))^2).$$

It is a measure of the spread of the possible values of X.

The standard deviation of X is the square root of the variance,

$$\sigma(X) = \sqrt{\text{var}(X)}.$$

It has the same units as $E(X)$.

Properties:

- $\text{var}(X) = E(X^2) - (E(X))^2$
- $\text{var}(aX + b) = a^2\text{var}(X)$

Independence of random variables

The random variables X and Y are independent if for all x, y,

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

or equivalently

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

Properties:

- If X and Y are independent, then so are $f(X)$ and $g(Y)$.
- If X and Y are independent, then

$$E(XY) = E(X)E(Y).$$

- If X and Y are independent, then

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y).$$

So: Variance of sum = sum of variance (only if independent)!

Independence of random variables

If $X_1, \ldots, X_n$ are independent, then

$$\text{var}(X_1 + \cdots + X_n) = \text{var}(X_1) + \cdots \text{var}(X_n)$$

and so, if $X_1, \ldots, X_n$ have the *same distribution*,

$$\sigma(X_1 + \cdots + X_n) = \sigma\sqrt{n}$$

and thus for the sample mean,

$$\sigma((X_1 + \cdots + X_n)/n) = \sigma/\sqrt{n}.$$

This is known as the square-root law.

Example:

What is the expectation and variance of $X - Y$, if X and Y are independent?

Variability pooling

- Company offers 100 configurations of a product, all having component A
- Mean demand per day for each configuration is 10, standard deviation 3

Option 1: Stock all configurations

- Base stock level is mean demand plus safety stock
- Safety stock is 2 times standard deviation of demand
- Stock level for each configuration is $10 + 2 \cdot 3 = 16$
- Total stock level of components A is $100 \cdot 16 = 1600$

Option 2: Stock only A and assemble to order

- Stock level for component A is $100 \cdot 10 + 2 \cdot \sqrt{100} \cdot 3 = 1060$

Conclusion:
Pooling leads to reduction of more than 30% in stock level of component A!

Convolution

Suppose the possible values of X and Y are $0, 1, 2, \ldots$.

Property: If X and Y are independent, then for $k = 0, 1, \ldots$

$$\begin{aligned} P(X+Y=k) &= \sum_{j=0}^{k} P(X=j, Y=k-j) \\ &= \sum_{j=0}^{k} P(X=j)P(Y=k-j). \end{aligned}$$

Discrete random variables

- Bernoulli random variable X with success probability p,

$$P(X = 1) = 1 - P(X = 0) = p.$$

Then

$$E(X) = p, \quad \text{var}(X) = p(1 - p).$$

- Binomial random variable X is the number of successes in n independent trials, each with probability p of success,

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, 1, \ldots n.$$

Then

$$E(X) = np, \quad \text{var}(X) = np(1 - p).$$

k out of n system

Example: k out of n system.

Consider a system composed of n identical components, and q is the probability that a component works (independent of the others). At least k components have to work for the system to work.

Then the probability Q that the system works is

$$Q = \sum_{i=k}^{n} \binom{n}{i} q^i (1-q)^{n-i}.$$

Discrete random variables

- Poisson random variable X with parameter $\lambda > 0$,

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

Then

$$E(X) = \text{var}(X) = \lambda.$$

Property: Poisson + Poisson = Poisson!

- Hypergeometric random variable X is number of red balls in a random selection of n balls from an urn with R red balls and W white balls,

$$P(X = r) = \frac{\binom{R}{r}\binom{W}{n-r}}{\binom{R+W}{n}}, \quad r = 0, 1, \ldots, n.$$

Then

$$E(X) = n \frac{R}{R + W}.$$

Coincidence problem

Two people, strangers to one another, both living in Eindhoven, meet each other. Each has approximately 200 acquaintences in Eindhoven.

What is the probability of the two people having an acquaintance in common?

Binomial and Poisson

Relation between Binomial and Poisson:

In a *very large* number n of independent trials, each with a *very small* probability p of success, the total number of successes is approximately Poisson distributed with expected value np.

Or more precisely: for $k = 0, 1, \ldots$,

$$\lim_{n\to\infty, p\to 0} \binom{n}{k} p^k (1-p)^{n-k} = e^{-\lambda} \frac{\lambda^k}{k!},$$

where $\lambda = np$ (which is assumed to remain constant).

Scratch-and-win lottery

Each week a very popular lottery in Andorra prints 10^4 tickets. Each tickets has two 4-digit numbers on it, one visible and the other covered. The numbers are randomly distributed over the tickets. If someone, after uncovering the hidden number, finds two identical numbers, he wins a large amount of money.

- What is the average number of winners per week?
- What is the probability of at least one winner?

The same lottery prints 10^7 tickets in Spain.
What about the answers to the questions above?

Scratch-and-win lottery

Simulation results for Spain based on 10^5 experiments

Average numbers of winners is 1.001.

k	$P(k$ winners$)$
0	0.367
1	0.370
2	0.183
3	0.061
4	0.016

Discrete random variables

- Geometric random variable X is number trials till first success, each trial with probability p of success,

$$P(X = k) = (1-p)^{k-1} p, \quad k = 1, 2, \ldots$$

Then

$$E(X) = \frac{1}{p}, \quad \text{var}(X) = \frac{1-p}{p^2}.$$

- Negative binomial random variable X is number trials till rth success, each trial with probability p of success,

$$P(X = k) = \binom{k-1}{r-1} (1-p)^{k-r} p^r, \quad k = r, r+1, \ldots$$

Then

$$E(X) = \frac{r}{p}, \quad \text{var}(X) = \frac{r(1-p)}{p^2}.$$

Inspection and rework

- Workstation performs tasks
- Task time is exactly 1 time unit (no variability)
- Task is checked after completion
- Task is done correctly with probability p
- If not, task is repeated (and checked till it is eventually correct)

X denotes total effective task time, so

$$P(X = k \text{ time units}) = (1-p)^{k-1}p, \quad k = 1, 2, \ldots$$

Then

$$E(X) = \frac{1}{p}, \quad \text{var}(X) = \frac{1-p}{p^2}.$$