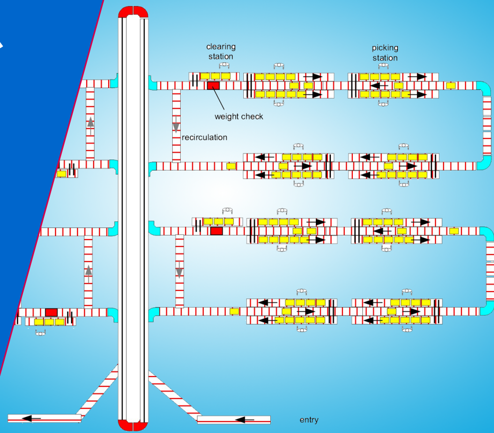
Analysis of Manufacturing Systems 4AB00

Ivo Adan



Technische Universiteit
Eindhoven
University of Technology

Contents

- Chapter 13: 13.1, 13.2, 13.3 (till 13.3.1)

Recap: Central limit theorem

$X_1, X_2, \ldots$ are independent random variables with the same distribution, and

$$\mu = E(X), \quad \sigma = \sigma(X)$$

Then

$$E(X_1 + \cdots + X_n) = n\mu, \quad \text{var}(X_1 + \cdots + X_n) = n\sigma^2$$

and

$X_1 + \cdots + X_n$ has approximately a normal distribution

with mean $n\mu$ and variance $n\sigma^2$ when n is large

Recap: Normal distribution

Example:

$Y = e^X$ where X is normal with parameters μ and σ. Density of Y?

Distribution of Y

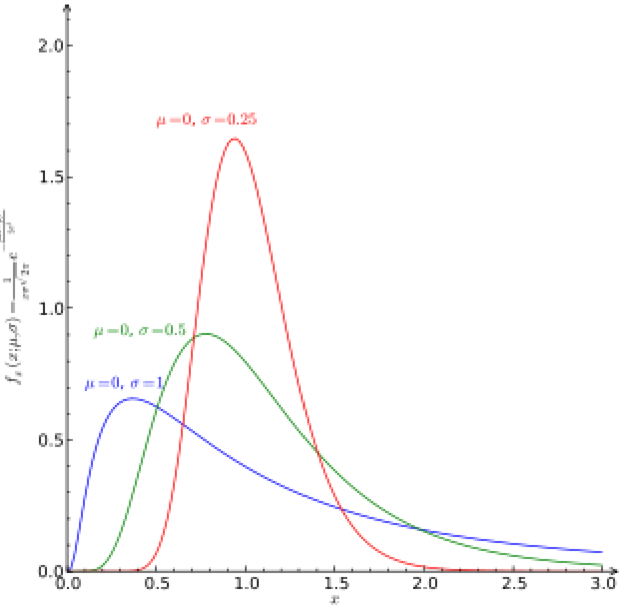
$$\begin{aligned} P(Y \le y) &= P(e^X \le y) \\ &= P(X \le \ln(y)) \\ &= \int_{-\infty}^{\ln(y)} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2} dx \\ &= \int_0^y \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\ln(t)-\mu)^2/\sigma^2} \frac{1}{t} dt \end{aligned}$$

So density of Y

$$f_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\ln(y)-\mu)^2/\sigma^2} \frac{1}{y}, \quad y > 0$$

This is lognormal density with parameters μ and σ

Recap: Lognormal density



Applications

- Income distribution
- Insurance claim sizes
- Service times at call centers

Recap: Lognormal density

Example:

Population of bacteria has initial size s_0.
Each generation is equally likely to increase by 25% or decrease by 20%.

What is approximate density of population size s_n of generation n for large n?

Recap: Joint random variables

- Joint probability distribution function of continuous variables X and Y

$$P(X \leq a, Y \leq b) = \int_{x=-\infty}^{a} \int_{y=-\infty}^{b} f(x, y) dx dy$$

- Joint density $f(x, y)$ satisfies

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} P(X \leq x, Y \leq y)$$

- Interpretation of joint density: for small $\Delta > 0$

$$P(x < X \leq x + \Delta, y < Y \leq y + \Delta) \approx f(x, y)\Delta^2$$

- X and Y are independent if

$$f(x, y) = f_X(x) f_Y(y) \quad \text{for all } x, y.$$

Production lines

Synchronous lines:

- Coordinated (simultaneous) movement of jobs
- Total WIP (number of jobs) is constant
- No buffers needed
- Paced (maximum limit for process time) or unpaced

Asynchronous lines:

- No coordination of movement of jobs
- Total WIP fluctuates
- Blocking and starvation of machines
- Buffers needed

Unpaced synchronous lines

- m machines in series
- X_i processing time of machine i with distribution F_i
- Process times are independent
- C is the cycle time, so $C = \max\{X_1, \ldots, X_m\}$

$$\begin{aligned} F_C(t) &= P(C \leq t) \\ &= P(\max\{X_1, \ldots, X_m\} \leq t) \\ &= P(X_1 \leq t, \ldots, X_m \leq t) \\ &= F_1(t) \cdots F_m(t) \end{aligned}$$

Then throughput δ of the line

$$\delta = \frac{1}{E(C)}$$

where $E(C) = \int_0^\infty t f_C(t) dt$

Unpaced synchronous lines

What can variability of processing times do to the throughput?

Examples:

- X_i are uniform on $(0, 1)$, then

$$E(C) = 1 - \frac{1}{m+1}$$

- X_i are exponential with rate 2, then

$$E(C) = \frac{1}{2}\left(\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{2} + 1\right) \approx \frac{1}{2}\log(m)$$

Paced synchronous lines

- m machines in series
- X_i processing time of machine i with distribution F_i
- Fixed cycle time c

Then throughput δ of jobs with no defects

$$\delta = \frac{Q(c)}{c}$$

where

$$\begin{aligned} Q(c) &= P(X_1 \leq c, \ldots, X_m \leq c) \\ &= P(X_1 \leq c) \cdots P(X_m \leq c) \\ &= F_1(c) \cdots F_m(c) \end{aligned}$$

So trade-off between volume δ of output and quality $Q(c)$ of output

Paced synchronous lines

Cycle time c^* maximizing δ is solution of

$$\frac{d}{dc}Q(c) = \frac{Q(c)}{c}$$

Cycle time c should never be set smaller than c^*! Why?

Example: X_i are exponential with rate 1

m	c	c^*	$Q(c^*)$	δ
5	5.51	2.55	0.66	0.26
10	6.21	3.60	0.76	0.21
20	6.90	4.50	0.80	0.18

c is minimal cycle time to meet $Q = 0.98$

Conditional probability mass

X and Y are discrete random variables with joint probability mass function

$$p(x, y) = P(X = x, Y = y).$$

Conditional probability mass function of X given Y is

$$P(X = x | Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}.$$

Note

$$P(X = x) = \sum_{y} P(X = x | Y = y) P(Y = y),$$

which is the law of conditional probability

Conditional probability mass

Example:

Simultaneously roll 24 dice and next roll with those that showed 6.
Let X be number of sixes in first roll and Y those in the second roll.

- What is $P(Y = y|X = x)$?
- And $P(X = x|Y = y)$?

Conditional probability density

X and Y are continuous random variables with joint density $f(x,y)$ and marginal densities $f_X(x)$ and $f_Y(y)$.

Conditional probability density of X given Y is

$$\begin{aligned} f_X(x|y)dx &= P(x < X \le x+dx | y < Y \le y+dy) \\ &= \frac{P(x < X \le x+dx, y < Y \le y+dy)}{P(y < Y \le y+dy)} \\ &= \frac{f(x,y)dxdy}{f_Y(y)dy} \\ &= \frac{f(x,y)}{f_Y(y)}dx, \end{aligned}$$

so

$$f_X(x|y) = \frac{f(x,y)}{f_Y(y)}$$

Conditional probability distribution

Conditional probability distribution of X given $Y = y$ is

$$P(X \leq x|Y = y) = \int_{-\infty}^{x} f_X(u|y)du$$

and law of conditional probability

$$P(X \leq x) = \int_{-\infty}^{\infty} P(X \leq x|Y = y) f_Y(y)dy$$

Example:

Point (X, Y) is randomly chosen in the unit circle.
What is the conditional density of X?

Law of conditional probability

For discrete random variables X and Y,

$$P(X = x) = \sum_y P(X = x | Y = y) P(Y = y)$$

For continuous random variables X and Y,

$$P(X \leq x) = \int_{-\infty}^{\infty} P(X \leq x | Y = y) f_Y(y) dy$$

Law of conditional probability

Example:

Mr. Johnson waits for the metro. Once the metro has stopped, the distance to the nearest metro door is uniform between 0 and 2 metres. He is able to find a place to sit with probability

$$1 - \sqrt{\frac{1}{2}y}$$

if the distance to the nearest door is y metres.

- What is the probability Mr Johnson finds a place to sit?

Law of conditional expectation

Conditional expectation of X given $Y = y$:

- Discrete random variables X and Y

$$E(X|Y = y) = \sum_x x P(X = x|Y = y),$$

and

$$E(X) = \sum_y E(X|Y = y) P(Y = y).$$

- Continuous random variables X and Y

$$E(X|Y = y) = \int_{-\infty}^{\infty} x f_X(x|y) dx,$$

and

$$E(X) = \int_{-\infty}^{\infty} E(X|Y = y) f_Y(y) dy.$$

Law of conditional expectation

Example:

Generate two random numbers from $(0, 1)$. Let X be the smallest and Y be the largest.

- What are $E(X|Y = y)$ and $E(Y|X = x)$?
- And what are $E(X)$ and $E(Y)$?
- Same questions, but now for three random numbers.

Example:

A batch consists of n items with probability $(1 - p)p^{n-1}$, $n \geq 1$.
The production time of a single item is uniform between 4 and 10 minutes.

- What is the mean production time of a batch?

Example:

The process time is exponential with rate Λ, where Λ is random with density

$$f_\Lambda(\lambda) = \lambda e^{-\frac{1}{2}\lambda^2}, \quad \lambda > 0$$

- What is the mean process time?