4 Transient analysis of Markov processes

In this chapter we develop methods of computing the transient distribution of a Markov process.

Let us consider an irreducible Markov process with finite state space $\{0, 1, \ldots, N\}$ and generator Q with elements q_{ij}, $i, j = 0, 1, \ldots, N$. The random variable $X(t)$ denotes the state at time t, $t \geq 0$. Then we want to compute the transition probabilities

$$p_{ij}(t) = P(X(t) = j | X(0) = i),$$

for each i and j. Once these probabilities are known, we can compute the distribution at time t, given the initial distribution, according to

$$P(X(t) = j) = \sum_{i=0}^{N} P(X(0) = i) p_{ij}(t), \qquad j = 0, 1, \ldots, N.$$

4.1 Differential equations

It is easily verified that for $t \geq 0$ the probabilities $p_{ij}(t)$ satisfy the differential equations

$$\frac{d}{dt} p_{ij}(t) = \sum_{k=0}^{N} p_{ik}(t) q_{kj}$$

or in matrix notation

$$\frac{d}{dt} P(t) = P(t) Q,$$

where $P(t)$ is the matrix of transition probabilities $p_{ij}(t)$. The above differential equations are refereed to as the *Kolmogorov's Forward Equations* (Clearly, there are also Backward Equations). The solution of this system of differential equations is given by

$$P(t) = \sum_{n=0}^{\infty} \frac{Q^n}{n!} t^n = e^{Qt}, \qquad t \geq 0.$$

Direct use of the infinite sum of powers of Q to compute $P(t)$ may be inefficient, since Q contains both positive and negative elements. An alternative is to reduce the infinite sum to a finite one by using the spectral decomposition of Q. Let $\lambda_i, 0 \leq i \leq N$, be the N eigenvalues of Q, assumed to be distinct, and let y_i and x_i be orthonormal left and right eigenvectors corresponding to λ_i. Further, let Λ be the diagonal matrix of eigenvalues, X the matrix of column vectors x_i and Y the matrix of row vectors y_i. Then we have

$$Q = X \Lambda Y, \qquad Y X = I,$$

and thus

$$\begin{aligned} P(t) &= e^{Qt} \\ &= \sum_{n=0}^{\infty} \frac{Q^n}{n!} t^n \\ &= \sum_{n=0}^{\infty} \frac{(X \Lambda Y)^n}{n!} t^n \\ &= \sum_{n=0}^{\infty} \frac{X \Lambda^n Y}{n!} t^n \\ &= X e^{\Lambda t} Y, \end{aligned}$$

where $e^{\Lambda t}$ is just the diagonal matrix with elements $e^{\lambda_i t}$. The difficulty with the above solution is that it requires the computation of eigenvalues and eigenvectors. In the next section we present a numerically stable solution based on uniformization.

4.2 Uniformization

To establish that the sojourn time in each state is exponential with the same mean, we introduce fictitious transitions. Let Δ satisfy

$$0 < \Delta \leq \min_i \frac{1}{-q_{ii}}.$$

In state i, $0 \leq i \leq N$, we now introduce a transition from state i to itself with rate $q_{ii} + 1/\Delta$. This implies that the total outgoing rate from state i is $1/\Delta$, which does not depend on i! Hence, transitions take place according to a Poisson process with rate $1/\Delta$, and the probability to make a transition from state i to j is given by

$$p_{ij} = \Delta q_{ij} + \delta_{ij}, \qquad 1 \leq i, j \leq N.$$

If we denote the matrix of transition probabilities by P and condition on the number of transitions in $(0, t)$, we immediately obtain

$$P(t) = \sum_{n=0}^{\infty} e^{-t/\Delta} \frac{(t/\Delta)^n}{n!} P^n. \tag{1}$$

Since this representation requires addition and multiplication of nonnegative numbers only, it is suitable for numerical calculations, where we approximate the infinite sum by using the first K terms. A good rule of thumb for choosing K is

$$K = \max\{20, t/\Delta + 5 \cdot \sqrt{t/\Delta}\}.$$

For a more elaborated algorithm we refer to [1]. It further makes sense to take the largest possible value of Δ (Why?), so

$$\Delta = \min_i \frac{1}{-q_{ii}}.$$

4.3 Occupancy times

In this section we concentrate on the occupancy time of a given state, i.e., the expected time in that state during the interval $(0, T)$.

Let $m_{ij}(T)$ denote the expected amount of time spent in state j during the interval $(0, T)$, staring in state i. Then we have

$$m_{ij}(T) = \int_{t=0}^{T} p_{ij}(t)dt,$$

or in matrix notation,

$$M(T) = \int_{t=0}^{T} P(t)dt,$$

where $M(T)$ is the matrix with elements $m_{ij}(T)$. Substitution of (1) leads to

$$M(T) = \sum_{n=0}^{\infty} \int_{t=0}^{T} e^{-t/\Delta} \frac{(t/\Delta)^n}{n!} dt P^n.$$

By partial integration we get

$$\int_{t=0}^{T} e^{-t/\Delta} \frac{(t/\Delta)^n}{n!} dt = \Delta \left(1 - \sum_{k=0}^{n} e^{-T/\Delta} \frac{(T/\Delta)^k}{k!} \right) = \Delta P(Y > n),$$

where Y is a Poisson random variable with mean T/Δ. Hence, we finally obtain

$$M(T) = \Delta \sum_{n=0}^{\infty} P(Y > n) P^n.$$

The above representation provides a stable method for the computation of $M(T)$; for more details see [1].

4.4 Exercises

Exercise 1.
Consider a machine shop consisting of N identical machines and M repair men ($M \leq N$). The up times of the machines are exponential with mean $1/\mu$. When a machine fails, it is repaired by a repair man. The repair times are exponential with mean $1/\lambda$.

(i) Model the repair shop as a Markov process.

Suppose $N = 4$, $M = 2$, the mean up time is 3 days and the mean repair time is 2 hours. At 8:00 a.m. all machines are operating.

(ii) What is the expected number of working machines at 5:00 p.m.?

(iii) What is the expected amount of time all machines are working from 8:00 a.m. till 5:00 p.m.?

References

[1] V.G. Kulkarni, *Modeling, analysis, design, and control of stochastic systems*, Springer, New York, 1999.