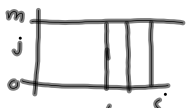
MC with states $\{\overbrace{(0,0),\dots,(0,m)}^{\text{level 0}},\overbrace{(1,0),\dots}^{\text{level 1}}\}$



$$P = \begin{pmatrix} B_0 & A_0 & 0 & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix}$$

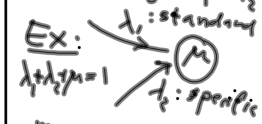
A_i : m+1 x m+1 matrix

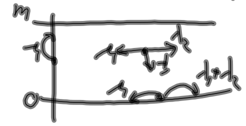
dec 10-13:16

Assume P irreducible

Also $A_0+A_1+A_2$ has 1 recurrent class

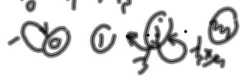
Ex:





(i,j): i = # orders in system
j = # standard products on stock

$A_0+A_1+A_2$:



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Stability: π eq. distr of $A_0+A_1+A_2$

P is ergodic iff $\pi A_0 e < \pi A_2 e$

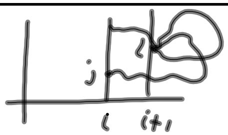
or: $\pi\cdot[\underbrace{1\cdot A_0 e - A_2 e}_{\text{mean step size}}] < 0$

Let $p(i,j)$ be eq. prob. of (i,j)

and $P_i = (p(i,0),\dots,p(i,m))$

dec 10-13:21

Define



$R(j,l)$ = mean # visits to $(i+1,l)$ during excursion that starts in (i,j) and end when level i is visited for first time again

Then: $P_{i+1} = P_i R = \dots = P_0 R^i$, $i=0,1,\dots$

dec 10-13:32

P_0 follows from:

$P_0 = P_0 B_0 + P_1 A_2$

$+ \sum_{i=0}^{\infty} P_i e = 1 = P_0 (I-R)^{-1}$

How to find R?

dec 10-13:35

Def: $R(j,l)^{(k)}$ = mean # visits to $(i+k,l)$ ....

so: $R^{(1)} = R$, $R^{(k)} = R^{(k-1)} \cdot R^{(1)} = \dots = (R^{(1)})^k$

and: $R^{(1)} = A_0 + R^{(1)} \cdot A_1 + R^{(2)} \cdot A_2$

$= A_0 + R^{(1)} A_1 + (R^{(1)})^2 A_2$

so R is solution of

$$R = A_0 + R A_1 + R^2 A_2$$

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Let X be the minimal non negative solution of $X = A_0 + XA_1 + X^2A_2$

Determine X by iteration:

$X_0 = 0$, $X_{k+1} = A_0 + X_kA_1 + X_k^2A_2$

Then $X_k \uparrow$, $X_k \le R$

Clearly: $X \le R$.

Claim: $R \le X$

dec 10-13:40

Def: $R_n^{(k)}(j,l)$ = mean # visits to $(i+k, l)$ ....

end when first visit to level i or n transitions have been made.

Thm: $R_n^{(k)} \le R_n^{(k-1)} R_n^{(1)} \le \cdots \le (R_n^{(1)})^k$

and: $R_n^{(1)} \uparrow R^{(1)}$, $n \to \infty$

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Then: $R_0^{(1)} = 0 \le X_0$

Assume: $R_n^{(1)} \le X_n$

then: $R_{n+1}^{(1)} = A_0 + R_n^{(1)}A_1 + R_n^{(2)}A_2$

$\le A_0 + R_n^{(1)}A_1 + (R_n^{(1)})^2A_2$

$\le A_0 + X_nA_1 + X_n^2A_2 = X_{n+1}$

Hence: $R = R^{(1)} = \lim_{n\to\infty} R_n^{(1)} \le \lim_{n\to\infty} X_n = X$

dec 10-13:47

Matrix-geom:

$P_i = P_0 R^i$, $i = 0, 1, \ldots$

where R is min. nonneg. solut. of $R = A_0 + RA_1 + R^2A_2$

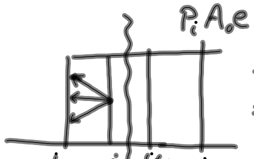
Special cases:

a) $A_2 = V\cdot\alpha = \begin{pmatrix} v_0 \\ \vdots \\ v_m \end{pmatrix} (\alpha_0 \cdots \alpha_m) = \begin{pmatrix} v_0\alpha \\ v_1\alpha \\ \vdots \\ v_m\alpha \end{pmatrix}$

with $\alpha e = 1$

dec 10-13:50

so:



$P_iA_0e = P_{i+1}A_2e$

$= P_{i+1} v \alpha e$

$= P_{i+1} v$

$R = A_0 + RA_1 + R^2 A_2$

$RA_0e\cdot\alpha$

so: $R = A_0(I - A_1 - A_0e\alpha)^{-1}$

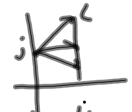
or: $P_i = P_{i-1}A_0 + P_iA_1 + P_{i+1}A_2 = P_iA_0e\cdot\alpha$

$A_2 = V\cdot\alpha$

dec 10-14:04

b). $A_0 = w\cdot\beta = \begin{pmatrix} w_0\beta \\ \vdots \\ w_m\beta \end{pmatrix}$

$\beta = (\beta_0, \ldots, \beta_m)$. $\beta e = 1$



Thm: $R = \begin{pmatrix} w_0\gamma \\ \vdots \\ w_m\gamma \end{pmatrix}$, $\gamma = (\gamma_0, \ldots, \gamma_m)$

$R^2 = w\gamma w\gamma$

$= \rho R$

$R^k = \rho^{k-1} R$

dec 10-14:12

So: $P_i = P_0 R^i$

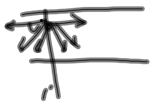
$= P_0 \rho^{i-1} R$

$= \rho^{i-1} P_0 R$

c) A_0, A_1, A_2: $\begin{pmatrix} \ddots & 0 \\ & \end{pmatrix}$ $A_k(i,j) = 0$ if $j > i$.

This implies:

$R(j,l) = 0$ if $l > j$



dec 10-14:17

Spectral exp.

$$P_i = \sum_{k=0}^{m} c_k \, y_k \, x_k^i$$

where $x_0, \dots, x_m$ roots with $|x| < 1$

of: $\det(A_0 + xA_1 + x^2 A_2 - xI) = 0$

and $y_0, \dots, y_m$ are non-null solution of

$$y_k (A_0 + x_k A_1 + x_k^2 A_2 - x_k I) = 0$$

dec 10-14:22

GF: $P(z) = \sum_{i=0}^{\infty} P_i z^i$, $A(z) = A_0 + A_1 z + A_2 z^2$

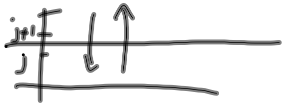
$\sum_{i=1}^{\infty} z^i P_i = z^i P_{i-1} A_0 + z^i P_i A_1 + z^i P_{i+1} A_2$, $i = 1, 2, \dots$

$\rightarrow P(z)[zI - A(z)] = P_0[z(B_0 - A_1) - A_0]$

Let $z_0, z_1, \dots, z_m$ roots of $\det(zI - A(z)) = 0$ with $|z| < 1$

and let $v_0, v_1, \dots, v_m$ be non-null solution of $(z_k I - A(z_k)) v_k = 0$

dec 10-14:27



Then:

$P_0(z_k(B_0 - A_1) - A_0) v_k = 0$, $k = 1, \dots, m$

diff, set $z = 1$, post multiply by e

? $P(1)[I - A'(1)]e = P_0[\underbrace{I - A_1 - (I - B_0)}_{B_0 - A_1}]e$

and: $P(1) = P(1) \cdot A(1) + P_0(B_0 + A_0 - A(1))$

$P(1)e = 1$

dec 10-14:32

Matrix G.

$G(j,l)$ = prob. of returning to level i in (i,l) when starting in $(i+1,j)$

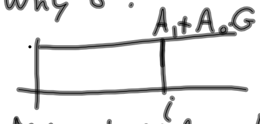
G satisfies:

$$G = A_0 + A_1 G + A_2 G^2$$

Note: $A_0 G = R A_2$

dec 10-14:33

Why G?



MC embedded on level $0, 1, \dots, i$

$$\tilde{P} = \begin{pmatrix} B_0 & A_0 & & 0 \\ A_2 & A_1 & A_0 & \\ & & \ddots & \\ 0 & & A_2 & A_1 + A_0 G \end{pmatrix}$$

dec 10-14:42

recursively comp. $P_0, P_1, \dots$

$P_0 = P_0 [B_0 + A_0 G]$

+ normalization eq. for P_0

Once $P_0, \dots, P_{i-1}$ are known:

$P_i = P_{i-1} A_0 + P_i [A_1 + A_2 G]$

dec 10-14:44

Now: G/M|1-type.

$$P = \begin{pmatrix} B_0 & A_0 & & & \\ B_1 & A_1 & A_0 & & 0 \\ B_2 & A_2 & A_1 & A_0 & \\ \vdots & & & & \\ \vdots & & & & \end{pmatrix}$$

dec 10-14:48

1) Matrix-geom. solution.

$P_i = P_0 R^i$

where R satisfies

$R = A_0 + RA_1 + R^2 A_2 + R^3 A_3 + \dots$

2). Spectral-exp: ✓.

3) GF ✗

dec 10-14:49

M|G|1-type.

$$P = \begin{pmatrix} B_0 & B_1 & B_2 & \dots & \\ A_0 & A_1 & A_2 & \dots & \\ 0 & A_0 & A_1 & A_2 & \dots \\ & & \ddots & & \ddots \end{pmatrix}$$

dec 10-14:51

G/G/1 ($\underset{PH}{G}$)

1) Matrix-Geom. ✗

2) Spectral exp. ✗

3) GF: works.

4) matrix G: works.

$G = A_0 + A_1 G + A_2 G^2 + A_3 G^3 + \dots$

dec 10-14:52