FITTING DISCRETE DISTRIBUTIONS ON THE FIRST TWO MOMENTS

IVO ADAN, MICHEL VAN EENIGE AND JACQUES RESING*

Abstract. In this paper we present a simple method to fit a discrete distribution on the first two moments of a given random variable. With the fitted distribution we solve approximately Lindley's equation for the discrete-time $D/G/1$ queue using a moment-iteration method. Numerical results show excellent performance of the method.

Key words. two-moment fits, discrete distributions, Lindley's equation, moment-iteration method

AMS subject classifications. 60K25, 90B22

1. Introduction. A common way to fit a continuous distribution on the mean, EX, and the coefficient of variation, c_X, of a given non-negative random variable is the following (see e.g. Tijms [9]). In case $0 < c_X < 1$ one fits a mixture of Erlangian distributions with the same scale parameter. More specifically, if $1/k \leq c_X^2 \leq 1/(k-1)$, for certain $k = 2, 3, \ldots$, one fits an $E_{k-1,k}$ distribution. A random variable having this distribution is with probability q (respectively $1-q$) distributed as the sum of $k-1$ (respectively k) independent exponentials with common mean. In case $c_X \geq 1$ one fits a hyperexponential distribution with balanced means. A random variable having this distribution is with probability q_1 (respectively q_2) distributed as an exponential with parameter μ_1 (respectively μ_2) where $q_1 + q_2 = 1$ and $q_1/\mu_1 = q_2/\mu_2$.

Many applications involve *discrete* random variables. In such cases it is, of course, more natural to fit discrete distributions instead of continuous ones. The aim of this paper is to construct a discrete analogue of the method described above. It turns out that this problem is not trivial. In fact, in order to fit all possible values of EX and c_X, we need four classes of distributions instead of only two. Ord [8] treats a problem related to the present one. However, Ord fits discrete distributions not on the first two moments, but on two parameters depending on the first three moments of a given random variable.

After the construction of the discrete fits on the first two moments, we use these fits for approximating the waiting-time characteristics of the discrete-time $D/G/1$ queue. This queueing model naturally arises, for example, in the analysis of a periodic review (R, S) inventory system and in the analysis of a fixed-cycle traffic light queue. We use a discrete version of the moment-iteration method of de Kok [5], originally introduced for the continuous-time case. Furthermore, we will check the quality of this method by comparing it, for several service-time distributions, with an exact numerical analysis, with Fredericks' approximation (see Fredericks [4]), and with its continuous-time version.

The rest of the paper is organized as follows. In Section 2 we answer the question how to fit a discrete distribution on the first two moments of a given random variable. Next, in Section 3 we describe the discrete version of de Kok's iteration method for approximating the waiting-time characteristics of the discrete-time $D/G/1$ queue. Finally, Section 4 is devoted to numerical results.

* Eindhoven University of Technology, Department of Mathematics and Computing Science, P.O.Box 513, 5600 MB - Eindhoven, The Netherlands.

2. Fitting discrete distributions. Consider an arbitrary pair of non-negative, real numbers (EX, c_X). Before we come to the issue of how to fit a distribution with mean EX and coefficient of variation c_X, let us first answer the question which combinations (EX, c_X) are possible for discrete distributions on the non-negative integers. Clearly, for continuous distributions on the non-negative real numbers, all combinations (EX, c_X) with $EX \geq 0$ and $c_X \geq 0$ are possible. However, for discrete distributions this turns out to be not the case.

LEMMA 2.1. *For a pair of non-negative, real numbers (EX, c_X), there exists a random variable X on the non-negative integers with mean EX and coefficient of variation c_X if and only if*

$$c_X^2 \geq \frac{2k+1}{EX} - \frac{k(k+1)}{(EX)^2} - 1, \tag{1}$$

where k is the unique integer satisfying $k \leq EX < k+1$.

Proof. For a given value of EX, there exists a unique random variable, concentrated on the integers k and $k+1$ only, with mean EX. For this random variable we have equality in (1) and its coefficient of variation is less than that of any other integer-valued random variable with mean EX. Hence (1) is a necessary condition. That (1) is also a sufficient condition is easily checked. □

If we define $a := c_X^2 - 1/EX$, then it follows from Lemma 2.1 that for all random variables on the non-negative integers $a \geq -1$. Lemma 2.2 shows how to fit a distribution with mean EX and coefficient of variation c_X depending on the value of a. We use four classes of distributions: Poisson, mixtures of binomial, mixtures of negative-binomial and mixtures of geometric distributions with balanced means.

Let us first introduce some notation. A $GEO(p)$ random variable has probability distribution $p_i = (1-p)p^i$, $i = 0, 1, 2, \ldots$, and an $NB(k, p)$ variable is the sum of k independent $GEO(p)$ variables. A $POIS(\lambda)$ random variable is Poisson distributed with mean λ, and a $BIN(k, p)$ variable is binomially distributed, where k is the number of trials and p the success probability.

LEMMA 2.2. *Let X be a random variable on the non-negative integers with mean EX and coefficient of variation c_X and let $a = c_X^2 - 1/EX$. Then the random variable Y matches the first two moments of X if Y is chosen as follows:*

1. *If $-1/k \leq a \leq -1/(k+1)$ for certain $k = 1, 2, 3, \ldots$, then*

$$Y = \begin{cases} BIN(k, p) & w.p.\ q, \\ BIN(k+1, p) & w.p.\ 1-q, \end{cases}$$

where

$$q = \frac{1 + a(1+k) + \sqrt{-ak(1+k) - k}}{1+a}, \quad p = \frac{EX}{k+1-q}.$$

2. *If $a = 0$, then $Y = POIS(\lambda)$ with $\lambda = EX$.*
3. *If $1/(k+1) \leq a \leq 1/k$ for certain $k = 1, 2, 3, \ldots$, then*

$$Y = \begin{cases} NB(k, p) & w.p.\ q, \\ NB(k+1, p) & w.p.\ 1-q, \end{cases}$$

where

$$q = \frac{(1+k)a - \sqrt{(1+k)(1-ak)}}{1+a}, \quad p = \frac{EX}{k+1-q+EX}.$$

4. If $a \geq 1$, then

$$Y = \begin{cases} GEO(p_1) & w.p.\ q_1, \\ GEO(p_2) & w.p.\ q_2, \end{cases}$$

where

$$p_1 = \frac{EX[1 + a + \sqrt{a^2 - 1}]}{2 + EX[1 + a + \sqrt{a^2 - 1}]}, \quad q_1 = \frac{1}{1 + a + \sqrt{a^2 - 1}},$$
$$p_2 = \frac{EX[1 + a - \sqrt{a^2 - 1}]}{2 + EX[1 + a - \sqrt{a^2 - 1}]}, \quad q_2 = \frac{1}{1 + a - \sqrt{a^2 - 1}}.$$

It is straightforward to check that the given distributions indeed have the same mean and coefficient of variation as X. Therefore, the proof of Lemma 2.2 is omitted. The fact that $p \leq 1$ in case 1 of this lemma is a consequence of equation (1). The results of the two lemmas above are illustrated in Figure 1.

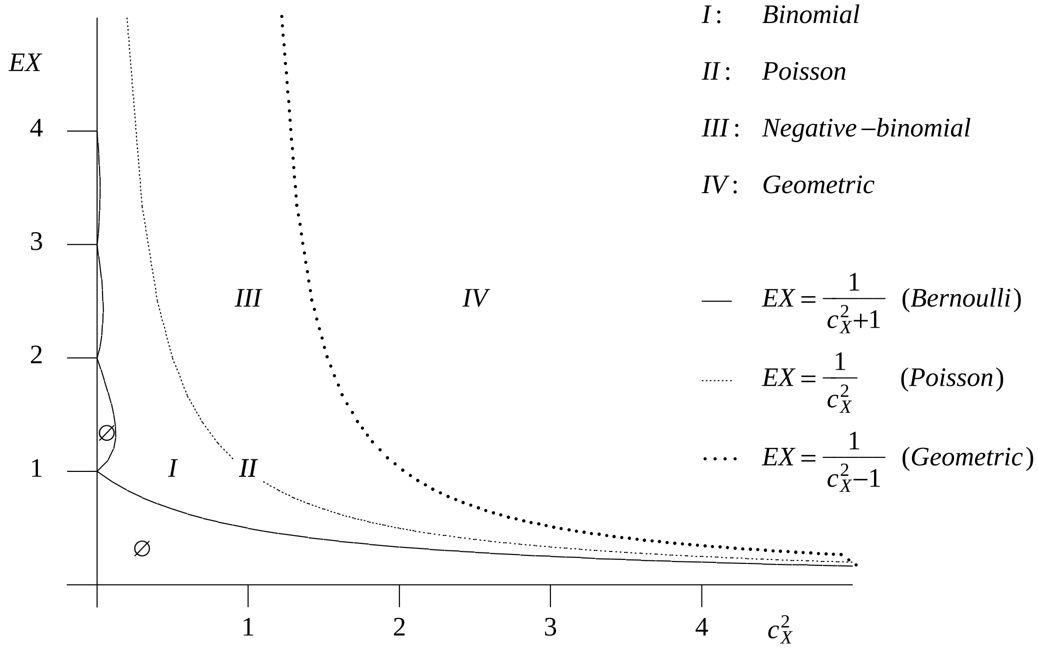

FIG. 1. *The four classes of distributions used to match the first two moments of X.*

Note: If $a > 0$ (cases 3 and 4 in Lemma 2.2), then it is also possible to fit an $NB(k,p)$ distribution with *real-valued* k. However, an advantage of the solution proposed in Lemma 2.2 is that the fitted distributions allow a simple interpretation in terms of sums or mixtures of geometric distributions.

3. Moment-iteration method. In this section, we describe a discrete version of de Kok's moment-iteration method for approximating waiting-time characteristics of the discrete-time $D/G/1$ queue. The present version is a refinement of the one in [5] in the sense that in the iteration step a distribution is fitted on the first two moments of the conditional waiting time instead of the unconditional waiting time plus the service time. Our interest in the discrete-time $D/G/1$ queue arose out of work on the fixed-cycle traffic light (see e.g. Darroch [2] and Newell [7]), and this queue also arises, for example, in the analysis of a periodic review (R, S) inventory system with finite production capacity (see e.g. de Kok [5]). For this reason we restricted our attention to deterministic interarrival times, but of course the method can easily be extended to more general interarrival-time distributions.

The waiting-time characteristics considered here are the delay probability Π_W, and the first two moments of the waiting time. The moment-iteration method is used in the next section to illustrate the method of fitting discrete distributions as presented in the previous section.

Consider a discrete-time $D/G/1$ queue. Customers arrive at the server with fixed interarrival time A. The service-time probability distribution is given by $\{b_i\}_{i=0}^{\infty}$ with mean b. The service times of customers are assumed to be mutually independent and customers are served in order of arrival. Furthermore, we assume that the queue is stable, i.e., $\rho := b/A < 1$.

Let, for $n = 0, 1, 2, \ldots$, the random variables B_n and W_n denote the service time, respectively, the waiting time of the n-th customer. Further, suppose that the 0-th customer arrives at an empty system, so that $W_0 = 0$. Then, it is easily seen that the following relation holds

$$W_n = \max\{0, W_{n-1} + B_{n-1} - A\}, \quad n = 1, 2, 3, \ldots.$$

This relation is the starting point for the approximation of the delay probability and the first two moments of the waiting-time distribution. Let the generic random variable B have probability distribution $\{b_i\}_{i=0}^{\infty}$, then the moment-iteration method (for the discrete-time $D/G/1$ queue) can be described as follows:

Moment-iteration algorithm for the discrete-time $D/G/1$ queue.

1. Initialization.

Set $W_0 := 0$, and compute exactly

$$\begin{aligned} P(W_1 > 0) &= P(B - A > 0), \\ EW_1 &= E(B - A)_+, \\ EW_1^2 &= E(B - A)_+^2, \end{aligned}$$

and set $n := 1$.

2. Iteration.

Set $V_n := (W_n | W_n > 0) - 1$, and compute

$$\begin{aligned} EV_n &= \frac{EW_n}{P(W_n > 0)} - 1, \\ EV_n^2 &= \frac{EW_n^2}{P(W_n > 0)} - \frac{2EW_n}{P(W_n > 0)} + 1. \end{aligned}$$

Fit a tractable probability distribution $\{v_i\}_{i=0}^{\infty}$ to the probability distribution of V_n by matching the first two moments as described in the previous section. Compute

$$\begin{aligned} EW_{n+1} &= E(B - A)_+ P(W_n = 0) + E(1 + V_n + B - A)_+ P(W_n > 0) \\ &= P(W_n = 0) \sum_{i=A+1}^{\infty} b_i (i - A) + P(W_n > 0) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} v_i b_j (1 + i + j - A)_+, \end{aligned} \tag{2}$$

and also

$$P(W_{n+1} > 0) = P(W_n = 0)P(B - A > 0) + P(W_n > 0)P(1 + V_n + B - A > 0),$$
$$EW_{n+1}^2 = E(B - A)_+^2 P(W_n = 0) + E(1 + V_n + B - A)_+^2 P(W_n > 0),$$

where the right-hand sides can be worked out similarly as in (2).
3. Convergence.
If it holds that

$$|EW_{n+1} - EW_n| < \epsilon_1 \quad \text{and} \quad |EW_{n+1}^2 - EW_n^2| < \epsilon_2,$$

then stop, otherwise set $n := n + 1$ and repeat step 2.
4. Stop.
Approximate Π_W by $P(W_{n+1} > 0)$, and the first two moments of the waiting-time distribution by EW_{n+1} and EW_{n+1}^2, respectively.

In the next section, this algorithm will be applied to the $D/G/1$ queue for several discrete service-time distributions.

4. Numerical results. In this section we present numerical results to illustrate the quality of the approximations obtained with the moment-iteration method. In Table 1 we list the moment approximations (mit) and the exact values (exa) of the delay probability Π_W and the mean waiting time EW for $D/G/1$ queues with binomial, negative-binomial and uniform service times, respectively. The performance characteristics Π_W and EW are evaluated for a range of values of the systems parameters. We also compared the moment approximation with the well-known approximation of Fredericks (Fred) for the $D/G/1$ queue (see [4]). The results in Table 1 are based on Fredericks' approximation for discrete service-time distributions, which can easily be derived from the continuous version in [4]. The exact results for the $D/BIN(k,p)/1$ queue have been obtained with use of the matrix-geometric approach developed by Neuts [6] (see also [3]). The $D/NB(k,p)/1$ queue has been analyzed exactly by using an embedded Markov chain approach similar as in [1] and the exact results for the $D/U(0,k)/1$ have been obtained by numerically solving the embedded Markov chain using the approach of Tijms and van de Coevering [10], where $U(0,k)$ denotes the uniform distribution on $0, 1, \ldots, k$.

The results in Table 1 show good performance of Fredericks' approximation and excellent performance of the moment approximation. Fredericks' approximation is exact for the $D/GEO(p)/1$ queue and the $D/BIN(k,p)/1$ and $D/U(0,k)/1$ queue with $k = A + 1$, since the waiting-time distribution for these queues is geometric.

Since we fit a tractable distribution to the distribution of $W_n | W_n > 0$, we are also able to compute an approximation for $P(W_{n+1} > i)$ for all $i = 1, 2, \ldots$ in the same way as for $P(W_{n+1} > 0)$. From this, we can compute approximations for the α-percentiles of the waiting-time distribution. For each α with $0 \le \alpha \le 1$ the α-percentile is defined as the smallest i satisfying $P(W_{n+1} \le i) \ge \alpha$. In Table 2 we list the moment approximations and the exact values of the α-percentiles for the $D/G/1$ queues of Table 1 with traffic load 0.9 and 0.95, where α is varied from 0.5, 0.9, 0.95 to 0.99. We see in Table 2 that the quality of the moment approximation is good.

TABLE 1
The delay probability Π_W and the mean waiting time EW.

	A	k	p	ρ	Π_W exa	Π_W mit	Π_W Fred	EW exa	EW mit	EW Fred
$D/BIN(k,p)/1$	1	2	0.25	0.5	0.1111	0.1111	0.1111	0.1250	0.1250	0.1250
	1	2	0.45	0.9	0.6694	0.6694	0.6694	2.0250	2.0243	2.0250
	1	2	0.475	0.95	0.8186	0.8186	0.8186	4.5125	4.5099	4.5125
	2	10	0.1	0.5	0.1025	0.1025	0.0998	0.1371	0.1371	0.1338
	2	10	0.18	0.9	0.6885	0.6887	0.6802	3.0759	3.0756	3.0484
	2	10	0.19	0.95	0.8324	0.8326	0.8270	7.0413	7.0396	7.0080
	6	10	0.3	0.5	0.0112	0.0112	0.0111	0.0132	0.0132	0.0131
	6	10	0.54	0.9	0.4701	0.4704	0.4579	1.2498	1.2497	1.2270
	6	10	0.57	0.95	0.6873	0.6877	0.6762	3.1878	3.1873	3.1533
	30	40	0.6	0.8	0.0163	0.0163	0.0161	0.0260	0.0260	0.0257
	30	40	0.7125	0.95	0.3972	0.3975	0.3806	1.3393	1.3389	1.3028
	30	40	0.735	0.98	0.6941	0.6948	0.6772	4.9204	4.9188	4.8484
$D/NB(k,p)/1$	10	1	0.8333	0.5	0.2129	0.2129	0.2129	1.6232	1.6232	1.6232
	10	1	0.9	0.9	0.8075	0.8075	0.8075	41.942	41.941	41.942
	10	1	0.9048	0.95	0.9018	0.9018	0.9018	96.472	96.467	96.472
	10	5	0.5	0.5	0.0760	0.0711	0.0690	0.2256	0.2077	0.2028
	10	5	0.6429	0.9	0.7025	0.7006	0.6872	10.520	10.505	10.383
	10	5	0.6552	0.95	0.8433	0.8424	0.8337	25.279	25.268	25.114
$D/U(0,k)/1$	5	6	–	0.6	0.1727	0.1727	0.1727	0.2087	0.2087	0.2087
	5	8	–	0.8	0.5341	0.5362	0.5100	2.1274	2.1234	2.0814
	5	9	–	0.9	0.7535	0.7564	0.7289	6.8230	6.8194	6.7216
	10	15	–	0.75	0.4430	0.4439	0.4180	2.2113	2.2058	2.1545
	10	18	–	0.9	0.7586	0.7615	0.7290	12.312	12.300	12.107
	10	19	–	0.95	0.8765	0.8785	0.8574	30.302	30.326	30.060

TABLE 2
The α-percentiles of the waiting-time distribution.

				α-Percentiles								
				α	0.5		0.9		0.95		0.99	
	A	k	p		exa	mit	exa	mit	exa	mit	exa	mit
$D/BIN(k,p)/1$	1	2	0.45		1	1	5	5	7	7	11	11
	1	2	0.475		3	3	11	11	14	14	23	22
	2	10	0.18		2	2	8	8	11	11	17	17
	2	10	0.19		5	5	17	17	23	22	35	34
	6	10	0.54		0	0	4	4	5	5	8	8
	6	10	0.57		2	2	8	8	11	11	18	17
	30	40	0.7125		0	0	4	4	6	6	10	10
$D/NB(k,p)/1$	10	1	0.9		25	25	108	108	144	144	226	226
	10	1	0.9048		63	63	235	235	308	308	480	480
	10	5	0.6429		6	6	28	29	38	38	61	59
	10	5	0.6552		16	16	63	63	83	82	130	125
$D/U(0,k)/1$	5	9	–		4	4	17	17	23	22	36	33
	10	18	–		8	8	31	31	42	40	66	60
	10	19	–		20	22	73	72	96	92	150	136

Instead of using discrete distributions one may also fit a continuous distribution $W(t)$ on the first two moments of $W_n|W_n > 0$ as described in the introduction. Then equation (2) for EW_{n+1} becomes

$$EW_{n+1} = E(B-A)_+P(W_n = 0) + P(W_n > 0)\int_{t=0}^{\infty}\sum_{j=0}^{\infty} b_j(t+j-A)_+dW(t)$$

and the equations for $P(W_{n+1} > 0)$ and EW_{n+1}^2 have to be adapted accordingly. In Table 3 we compare for several examples the quality of discrete (disc) and continuous (cont) fits. The results show that both fits perform excellent for the mean waiting time, but the discrete fits yield a significantly better approximation for the delay probability.

Table 3
Comparison of the quality of discrete and continuous fits.

					Π_W			EW		
	A	k	p	ρ	exa	disc	cont	exa	disc	cont
$D/BIN(k,p)/1$	1	2	0.25	0.5	0.1111	0.1111	0.1484	0.1250	0.1250	0.1349
	1	2	0.475	0.95	0.8186	0.8186	0.8704	4.5125	4.5099	4.5595
	2	5	0.3	0.5	0.0815	0.0815	0.0988	0.0990	0.0990	0.1027
	2	5	0.57	0.95	0.8081	0.8083	0.8486	5.2642	5.2630	5.2941
	2	10	0.1	0.5	0.1025	0.1025	0.1233	0.1371	0.1371	0.1413
	2	10	0.19	0.95	0.8324	0.8326	0.8651	7.0413	7.0396	7.0672
$D/NB(k,p)/1$	10	1	0.8333	0.5	0.2129	0.2129	0.2198	1.6232	1.6232	1.6197
	10	1	0.9048	0.95	0.9018	0.9018	0.9049	96.472	96.467	96.464
	5	3	0.4545	0.5	0.1302	0.1284	0.1381	0.3334	0.3283	0.3283
	5	3	0.6129	0.95	0.8689	0.8688	0.8795	23.112	23.108	23.112
$D/U(0,k)/1$	3	4	–	0.67	0.2757	0.2757	0.3076	0.3806	0.3806	0.3872
	3	5	–	0.83	0.5955	0.5979	0.6394	2.0802	2.0788	2.0893
	5	6	–	0.6	0.1727	0.1727	0.1859	0.2087	0.2087	0.2114
	5	9	–	0.9	0.7535	0.7564	0.7764	6.8230	6.8194	6.8211

Acknowledgment. The authors thank Ton de Kok for stimulating discussions.

REFERENCES

[1] I. Adan and Y. Zhao, *Analyzing $GI|E_r|1$ queues*, Memorandum COSOR 94-37, Eindhoven University of Technology, Dep. of Math. and Comp. Sci., 1994.
[2] J. N. Darroch, *On the traffic-light queue*, Ann. Math. Stat., 35 (1964), pp. 380–388.
[3] M. J.A. van Eenige, J. A.C. Resing and J. van der Wal, *A matrix-geometric analysis of queueing systems with periodic service interruptions*, Memorandum COSOR 93-32, Eindhoven University of Technology, Dep. of Math. and Comp. Sci., 1993.
[4] A. A. Fredericks, *A class of approximations for the waiting time distribution in a $GI/G/1$ queueing system*, Bell Syst. Techn. J., 61 (1982), pp. 295–325.
[5] A. G. de Kok, *A moment-iteration method for approximating the waiting-time characteristics of the $GI/G/1$ queue*, Prob. Engineer. Inform. Sci., 3 (1989), pp. 273–287.
[6] M. F. Neuts, *Matrix-geometric solutions in stochastic models*, Johns Hopkins University Press, Baltimore, 1981.
[7] G. F. Newell, *Queues for a fixed-cycle traffic light*, Ann. Math. Stat., 31 (1960), pp. 589–597.
[8] J. K. Ord, *On a system of discrete distributions*, Biometrika, 54 (1967), pp. 649–656.
[9] H. C. Tijms, *Stochastic modelling and analysis: a computational approach*, John Wiley & Sons, Chichester, 1986.
[10] H. C. Tijms and M. C.T. van de Coevering, *A simple numerical approach for infinite-state Markov chains*, Prob. Engineer. Inform. Sci., 5 (1991), pp. 285–295.