3 Markov processes

In this chapter we show that the problem of finding the equilibrium distribution of a Markov process is essentially the same as that for a Markov chain.

Let us consider an irreducible Markov process with finite state space $\{0, 1, \ldots, N\}$ and generator Q with elements q_{ij}, $i, j = 0, 1, \ldots, N$. Here q_{ij} with $i \neq j$ denotes the transition rate from state i to state j, and q_{ii} is defined as

$$q_{ii} = -\sum_{j \neq i} q_{ij}. \tag{1}$$

The equilibrium distribution $\{p_0, p_1, \ldots, p_N\}$ is characterized as the unique normalized solution to the system

$$p_i \sum_{j \neq i} q_{ij} = \sum_{j \neq i} p_j q_{ji}, \qquad i = 0, 1, \ldots, N,$$

or

$$\sum_{j=0}^{N} p_j q_{ji} = 0, \qquad i = 0, 1, \ldots, N.$$

In vector-matrix notation this reads as

$$pQ = 0,$$

where p is the equilibrium probability vector. The matrix Q is called the *generator* of the Markov process. Note that convention (1) implies that the row sums of Q are equal to zero, i.e., $Qe = 0$.

3.1 An equivalent Markov chain

The equilibrium distribution p may also be obtained from an *equivalent Markov chain* via an elementary transformation. Let Δ be any real number satisfying

$$0 < \Delta \leq \min_i \frac{1}{-q_{ii}}. \tag{2}$$

Then P defined as

$$P = I + \Delta Q$$

is a stochastic matrix (verify). The Markov chain with transition probability matrix P has exactly the same equilibrium distribution p as the original Markov process, since p satisfies $pQ = 0$ if and only if $pP = p(I + \Delta Q) = p$. Note that this transformation also works for infinite state Markov processes as long as $\sup_i -q_{ii}$ is finite.

The Markov chain P can be interpreted as the discretization of the Markov process Q (with time step Δ). Also, P can be seen as the Markov chain of jumps of a *uniformized* version of Q; this will be explained in the next section.

3.2 Uniformization

The transformation in the previous section also has a probabilistic interpretation. Let P be the Markov chain of jumps of Q; so the transition probabilities of P are $p_{ij} = -q_{ij}/q_{ii}$. Clearly, if the mean sojourn time $-1/q_{ii}$ in state i is the same for each i, then Q and P have exactly the same equilibrium distribution. The condition on q_{ii} seems restrictive, but it turns out that each Markov process can be put in this form by introducing fictitious transitions from a state i to itself, which is called *uniformization*.

Let Δ satisfy (2), and introduce a *fictitious* transition from state i to itself with rate $q_{ii} + 1/\Delta$. Clearly, this does not affect the equilibrium distribution of the Markov process Q. Now the total outgoing rate from state i is $q_{ii} + 1/\Delta - q_{ii} = 1/\Delta$, and thus the mean sojourn time in state i is Δ; now it is the same for all states i. Hence, the equilibrium distribution of the Markov process Q is the same as that of the Markov chain P of jumps, the transition probabilities of which are given by $p_{ij} = \Delta q_{ij} + \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.