5 Examples of $M/M/1$ type models

In this chapter we present some simple variations on the $M/M/1$ system. In the $M/M/1$ system customers arrive according to a Poisson process and the service times of the customers are independent and identically exponentially distributed. This system can be described by a Markov process with states i, where i is simply the number of customers in the system. The transition-rate diagram of the $M/M/1$ model is shown in figure 1, where λ is the arrival rate and μ the service rate.

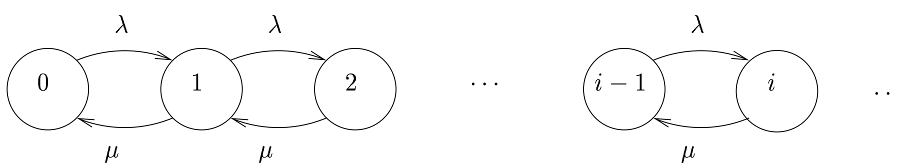

Figure 1: Transition-rate diagram for the $M/M/1$ model

Let p_i denote the (equilibrium) probability of state i, $i \geq 0$. From the transition-rate diagram it is easy to derive the equilibrium equations for the state probabilities p_i yielding

$$\begin{aligned} p_0\lambda &= p_1\mu, \\ p_i(\lambda+\mu) &= p_{i-1}\lambda + p_{i+1}\mu, \qquad i = 1, 2 \ldots, \end{aligned}$$

or by rearranging all terms at the same side of the equation,

$$\begin{aligned} -p_0\lambda + p_1\mu &= 0, && (1) \\ p_{i-1}\lambda - p_i(\lambda+\mu) + p_{i+1}\mu &= 0, \qquad i = 1, 2 \ldots && (2) \end{aligned}$$

Together with the normalization equation,

$$1 = \sum_{i=0}^{\infty} p_i,$$

this set of equations has a (unique) geometric solution

$$p_i = (1-\rho)\rho^i, \qquad i = 0, 1, 2, \ldots, \tag{3}$$

where $\rho = \lambda/\mu$.

An important feature of the system above is that transitions are restricted to neighboring states only, i.e., from state i to state $i-1$ or from state i to $i+1$. In the following sections we will consider models that share this feature, but in these models the simple state i is replaced by a set of states referred to as *level* i and the equilibrium distribution is a matrix generalization of (3); i.e., ρ will be replaced by a *rate matrix* R.

The excess probabilities for the waiting time W may be computed by conditioning on the state at arrival. Given that there are i customers in the system at arrival, the waiting time is Erlang-k distributed with mean k/μ. By PASTA, the probability of finding i customers at arrival is p_i. Hence, we get

$$\begin{aligned} P(W > t) &= \sum_{i=0}^{\infty}(1-\rho)\rho^i \sum_{j=0}^{i-1}\frac{(\mu t)^j}{j!}e^{-\mu t} = \sum_{j=0}^{\infty}\frac{(\mu t)^j}{j!}e^{-\mu t}\sum_{i=j+1}^{\infty}(1-\rho)\rho^i \\ &= \sum_{j=0}^{\infty}\frac{(\mu t)^j}{j!}e^{-\mu t}\rho^{j+1} = \rho e^{-\mu(1-\rho)t}, \qquad t \geq 0. \end{aligned} \tag{4}$$

5.1 Machine with set-up times

Let us consider a machine processing jobs in order of arrival. Jobs arrive according to a Poisson stream with rate λ and the processing times are exponential with mean $1/\mu$. For stability we assume that $\rho = \lambda/\mu < 1$. The machine is turned off when the system is empty and it is turned on again when a new job arrives. The set-up time is exponential with mean $1/\theta$. We are interested in the effect of the set-up time on the production lead time.

This model can be respresented as a Markov process with states (i,j) where i is the number of jobs in the system and j indicates whether the machine is on or off: $j=0$ means that the machine is off, $j=1$ means that it is on. The transition-rate diagram is displayed in figure 2. It looks similar to figure 1, except that each state i has been replaced by the set of states $\{(i,0),(i,1)\}$. This set of states is called level i. Transitions are now restricted to neigboring levels.

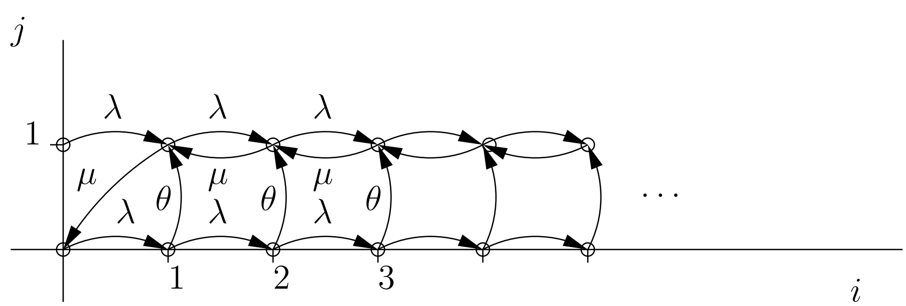

Figure 2: Transition-rate diagram for the $M/M/1$ model with set-up times

Let $p(i,j)$ denote the equilibrium probability of state (i,j), $i \geq 0$, $j=0,1$; clearly $p(0,1)=0$ since state $(0,1)$ is transient. From the transition-rate diagram we obtain by equating the flow out of a state and the flow into that state the following set of equilibrium equations,

$$p(0,0)\lambda = p(1,1)\mu, \tag{5}$$
$$p(i,0)(\lambda+\theta) = p(i-1,0)\lambda, \qquad i=1,2,\ldots \tag{6}$$
$$p(i,1)(\lambda+\mu) = p(i,0)\theta + p(i+1,1)\mu + p(i-1,1)\lambda, \qquad i=1,2,\ldots \tag{7}$$

The structure of the equations (6)-(7) is closely related to the similar set of equations (2). This becomes more striking by rewriting (6)-(7) in vector-matrix notation:

$$\begin{aligned} p_0 B_1 + p_1 B_2 &= 0, && (8) \\ p_{i-1} A_0 + p_i A_1 + p_{i+1} A_2 &= 0, \qquad i = 1, 2, \ldots && (9) \end{aligned}$$

where $p_i = (p(i,0), p(i,1))$ and

$$A_0 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad A_1 = \begin{pmatrix} -(\lambda+\theta) & \theta \\ 0 & -(\lambda+\mu) \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix}.$$

Obviously, if we can determine the equilibrium probabilities $p(i,j)$, then we can also compute the mean number of jobs in the system, and by Little's law, the mean production lead time. We now present two methods to determine the equilibrium probabilities. The first one is known as the *matrix-geometric method*, the other one is referred to as the *spectral expansion method*; see, e.g. [3, 1, 2]. Let us start with the matrix-geometric approach.

We first simplify the equilibrium equations (9) by eliminating the vector p_{i+1}. By equating the flow from level i to level $i+1$ to the flow from level $i+1$ to i (this is known as the *balance principle*) we obtain

$$(p(i,0) + p(i,1))\lambda = p(i+1,1)\mu$$

or in vector-matrix notation

$$p_i A_3 = p_{i+1} A_2$$

where

$$A_3 = \begin{pmatrix} 0 & \lambda \\ 0 & \lambda \end{pmatrix}.$$

Substituting this equation into (9) yields

$$p_{i-1} A_0 + p_i (A_1 + A_3) = 0, \qquad i > 1,$$

or

$$p_i = -p_{i-1} A_0 (A_1 + A_3)^{-1} = p_{i-1} R, \tag{10}$$

where

$$R = -A_0 (A_1 + A_3)^{-1} = \begin{pmatrix} \lambda/(\lambda+\theta) & \lambda/\mu \\ 0 & \lambda/\mu \end{pmatrix}.$$

Iterating (10) leads to the matrix-geometric solution

$$p_i = p_0 R^i, \qquad i = 0, 1, 2, \ldots. \tag{11}$$

Hence it is very similar to the solution for the $M/M/1$ model given by (cf. (3))

$$p_i = p_0 \rho^i, \qquad i = 0, 1, 2, \ldots.$$

Finally, p_0 follows from the equations (8) and the normalization equation

$$1 = \sum_{i,j} p(i,j) = p_0(I-R)^{-1}e,$$

where I is the identity matrix and e the column vector of ones. From (11) we obtain for $E(L)$, the mean number of jobs in the system,

$$E(L) = \sum_{i=1}^{\infty} i p_i e = \sum_{i=1}^{\infty} i p_0 R^i e = p_0 R(I-R)^{-2} e.$$

We now demonstrate the spectral expansion method. This method first seeks solutions of the equations (9) of the simple form

$$p_i = y \cdot x^i, \qquad i = 0, 1, 2, \ldots,$$

where $y = (y(0), y(1)) \neq 0$ and $|x| < 1$. The latter is required, since we want to be able to normalize the solution afterwards. Substitution of this form into (9) and dividing by common powers of x gives

$$y\left(A_0 + xA_1 + x^2 A_2\right) = 0.$$

Hence, the desired values of x are the roots inside the unit circle of the determinantal equation

$$\det(A_0 + xA_1 + x^2 A_2) = 0. \tag{12}$$

In this case we have

$$\det(A_0 + xA_1 + x^2 A_2) = (\lambda - (\lambda+\theta)x)(\mu x - \lambda)(x-1).$$

Hence, we find two roots, namely

$$x_1 = \frac{\lambda}{\lambda+\theta}, \qquad x_2 = \frac{\lambda}{\mu}.$$

For $i = 1, 2$, let y_i be a nonnull solution of

$$y\left(A_0 + x_i A_1 + x_i^2 A_2\right) = 0.$$

The final step of the spectral expansion method is to linearly combine the two simple solutions to also satify the boundary equations (8); note here that the equilibrium equations are linear. So we set

$$p_i = c_1 y_1 x_1^i + c_2 y_2 x_2^i, \qquad i = 0, 1, 2, \ldots \tag{13}$$

where the coefficients c_1 and c_2 follow from the boundary equations (8) and the normalization equation

$$1 = \frac{c_1 y_1 e}{1-x_1} + \frac{c_2 y_2 e}{1-x_2}.$$

Using representation (13) we obtain

$$E(L) = \sum_{i=1}^{\infty} i p_i e = \frac{c_1 y_1 x_1 e}{(1-x_1)^2} + \frac{c_2 y_2 x_2 e}{(1-x_2)^2}.$$

The two methods presented above are closely related: x_1 and x_2 are the eigenvalues of the rate matrix R and y_1 and y_2 are the corresponding eigenvectors.

Remark 5.1 The mean number of jobs in the system, $E(L)$, and the mean production lead time, $E(S)$, can also be determined by combining the PASTA property and Little's law. Based on PASTA we know that the average number of jobs in the system seen by an arriving job equals $E(L)$, and each of them (also the one being processed) has a (residual) processing time with mean $1/\mu$. With probability $1-\rho$ the machine is not in operation on arrival, so that the job also has to wait for the setup phase with mean $1/\theta$. Further, the job has to wait for its own processing time. Hence

$$E(S) = (1-\rho)\frac{1}{\theta} + E(L)\frac{1}{\mu} + \frac{1}{\mu},$$

and together with Little's law

$$E(L) = \lambda E(S),$$

we find

$$E(S) = \frac{1/\mu}{1-\rho} + \frac{1}{\theta}.$$

The first term at the right-hand side is the mean production lead time in the system without set-up times (i.e., the machine is always on). The second term is the mean set-up time. Clearly, the mean set-up time is exactly the extra mean delay caused by turning off the machine when there is no work. In fact, it can be shown (by using, e.g., a sample path argument) that the extra delay is an exponential time with mean $1/\theta$.

5.2 Unreliable machine

In this section we consider an unreliable machine processing jobs. The machine breaks down at random instants, whether it is processing a job or not. To obtain some insight in the effects of the breakdowns we study the following simple model.

Jobs arrive according to a Poisson stream with rate λ. The processing times are exponential with mean $1/\mu$. The up time of the machine is exponentially distributed with mean $1/\eta$. The repair time is also exponentially distributed with mean $1/\theta$.

This system can be described by a Markov process with states (i,j) where i is the number of jobs in the system and j indicates the state of the machine; the machine is up if $j=1$, it is down and in repair if $j=0$. The transition-rate diagram of this system is shown in figure 3. It again looks similar to figure 1, except that each state i has been replaced by the set of states $\{(i,0),(i,1)\}$.

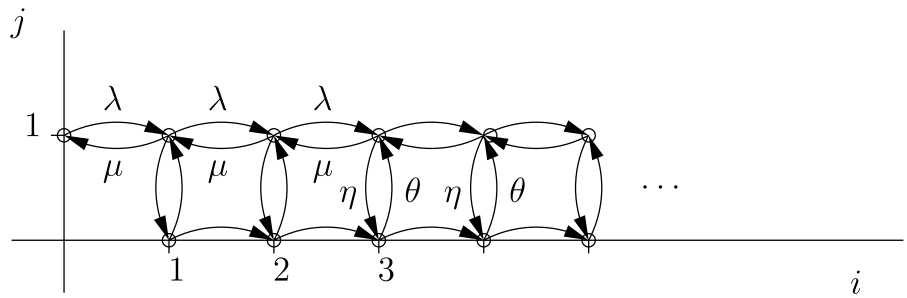

Figure 3: Transition-rate diagram for the $M/M/1$ model with random breakdowns

Let ρ_U denote the fraction of time the machine is up, so

$$\rho_U = \frac{1/\eta}{1/\eta + 1/\theta}.$$

Then, for stability, we have to require that

$$\frac{\lambda}{\mu} < \rho_U. \tag{14}$$

Let $p(i,j)$ denote the equilibrium probability of state (i,j). From the transition-rate diagram we get the following balance equations for the states $(0,0)$ and $(0,1)$,

$$\begin{aligned} p(0,0)(\lambda+\theta) &= p(0,1)\eta, && (15)\\ p(0,1)(\lambda+\eta) &= p(0,0)\theta + p(1,1)\mu, && (16)\end{aligned}$$

and for all states (i,j) with $i \geq 1$,

$$\begin{aligned} p(i,0)(\lambda+\theta) &= p(i-1,0)\lambda + p(i,1)\eta, \qquad i = 1,2,\ldots && (17)\\ p(i,1)(\lambda+\eta+\mu) &= p(i,0)\theta + p(i+1,1)\mu + p(i-1,1)\lambda, \quad i = 1,2,\ldots && (18)\end{aligned}$$

In vector-matrix notation these equations can be written as (cf. (9))

$$\begin{aligned} p_0 B_1 + p_1 A_2 &= 0, \\ p_{i-1}A_0 + p_i A_1 + p_{i+1}A_2 &= 0, \qquad i = 1,2,\ldots, \end{aligned}$$

where $p_i = (p(i,0), p(i,1))$ and

$$A_0 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad A_1 = \begin{pmatrix} -(\lambda+\theta) & \theta \\ \eta & -(\lambda+\mu+\eta) \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -(\lambda+\theta) & \theta \\ \eta & -(\lambda+\eta) \end{pmatrix}.$$

Similarly as for the $M/M/1$ model with set-up times we can show that the solution to these equations has a matrix-geometric form

$$p_i = p_0 R^i, \qquad i = 0, 1, 2, \ldots,$$

with

$$R = \frac{\lambda}{\mu} \begin{pmatrix} (\eta+\mu)/(\lambda+\theta) & 1 \\ \eta/(\lambda+\theta) & 1 \end{pmatrix},$$

or the following spectral expansion form

$$p_i = c_1 y_1 x_1^i + c_2 y_2 x_2^i, \qquad i = 0, 1, 2, \ldots$$

with x_1 and x_2 being the roots of

$$\mu(\lambda+\theta)x^2 - \lambda(\lambda+\mu+\eta+\theta)x + \lambda^2 = 0.$$

Based on these expressions for the equilibrium probabilities $p(i,j)$ it is easy to find closed-form expressions for the mean number of jobs in the system, $E(L)$, and the mean production lead time, $E(S)$.

5.3 The $M/E_r/1$ model

We consider a single-server queue. Customers arrive according to a Poisson process with rate λ and they are served in order of arrival. The service times are Erlang-r distributed with mean r/μ. For stability we require that the occupation rate

$$\rho = \lambda \cdot \frac{r}{\mu}$$

is less than one. This system can be described by a Markov process with states (i,j) where i is the number of customers waiting in the queue and j is the remaining number of service phases of the customer in service. The transition-rate diagram is shown in figure 4.

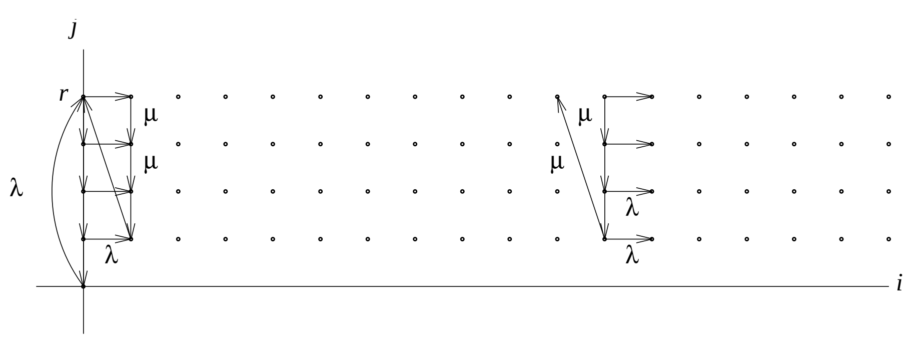

Figure 4: Transition-rate diagram for the $M/E_r/1$ model

Let $p(i,j)$ denote the equilibrium probability of state (i,j). From the transition-rate diagram we get the following balance equations for the states (i,j) with $i \geq 1$,

$$\begin{aligned} p(i,j)(\lambda+\mu) &= p(i-1,j)\lambda + p(i,j+1)\mu, \qquad j = 1,\dots,r-1, && (19)\\ p(i,r)(\lambda+\mu) &= p(i-1,r)\lambda + p(i+1,1)\mu, && (20)\end{aligned}$$

or in vector-matrix notation

$$p_{i-1}A_0 + p_iA_1 + p_{i+1}A_2 = 0, \qquad i \geq 1, \tag{21}$$

where $p_i = (p(i,1),\dots,p(i,r))$ and

$$A_0 = \begin{pmatrix} \lambda & 0 & \\ 0 & \ddots & 0 \\ & 0 & \lambda \end{pmatrix}, \quad A_1 = \begin{pmatrix} -(\lambda+\mu) & 0 & & \\ \mu & \ddots & 0 & \\ 0 & \ddots & \ddots & 0 \\ & 0 & \mu & -(\lambda+\mu) \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \cdots & 0 & \mu \\ \vdots & & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix}.$$

We first determine the probabilities $p(i,j)$ by using the matrix-geometric approach. Let level i denote the set of states $\{(i,1),\dots,(i,r)\}$. By balancing the flow between level i and level $i+1$ we get

$$(p(i,1)+\cdots+p(i,r))\lambda = p(i+1,1)\mu$$

or

$$p_iA_3 = p_{i+1}A_2, \tag{22}$$

where

$$A_3 = \begin{pmatrix} 0 & \cdots & 0 & \lambda \\ \vdots & & 0 & \lambda \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda \end{pmatrix}.$$

To eliminate p_{i+1} we substitute equation (22) into (21) yielding

$$p_{i-1}A_0 + p_i(A_1 + A_3) = 0.$$

Hence

$$p_i = p_{i-1}R,$$

where

$$R = -A_0(A_1+A_3)^{-1}.$$

Note that $A_1 + A_3$ is invertable, since it is a transient (or leak) generator. Iterating the above equation yields

$$p_i = p_0R^i, \qquad i = 0,1,2,\dots$$

Finally the probabilities $p(0,0)$ and p_0 follow from the equilibrium equations for the states $(0,0),\dots,(0,r)$ and the normalization equation.

To apply the spectral expansion method we substitute the simple form

$$p(i,j) = y(j)x^i, \qquad i = 0,1,2,\ldots, j = 1,\ldots,r,$$

into the equilibrium equations (19)-(20), yielding

$$\begin{aligned} y(j)x(\lambda+\mu) &= y(j)\lambda + y(j+1)x\mu, \qquad j = 1,\ldots,r-1, &(23)\\ y(r)x(\lambda+\mu) &= y(r)\lambda + y(1)x^2\mu. &(24)\end{aligned}$$

Hence

$$\frac{y(j+1)}{y(j)} = \frac{x(\lambda+\mu)-\lambda}{x\mu} = \text{constant} \equiv y,$$

so

$$y(j) = y^j, \qquad j = 1,\ldots,r.$$

Substituting this into (23)-(24) gives

$$\begin{aligned} x(\lambda+\mu) &= \lambda + yx\mu,\\ x(\lambda+\mu) &= \lambda + \frac{x^2\mu}{y^{r-1}}.\end{aligned}$$

This set of equations is equivalent to

$$\begin{aligned} x &= y^r, &(25)\\ y^r(\lambda+\mu) &= \lambda + y^{r+1}\mu. &(26)\end{aligned}$$

It can be shown that equation (26) has exactly r different (possibly complex) roots with $|y| < 1$; label these roots $y_1,\ldots,y_r$. Thus we find r *basis solutions* of the form

$$p(i,j) = y_k^j x_k^i, \qquad k = 1,\ldots,r,$$

where $x_k = y_k^r$. The next step is to take a linear combination of these basis solutions; so we set

$$p(i,j) = \sum_{k=1}^{r} c_k y_k^j x_k^i, \qquad i = 0,1,2,\ldots, j = 1,\ldots,r,$$

and determine the coefficients $c_1,\ldots,c_k$ and $p(0,0)$ such thate the equilibrium equations for the states $(0,j)$, $0 \le j \le r$ and the normalization equation are satisfied.

The excess probabilities for the waiting time W may be computed in exactly the same way as for the $M/M/1$ model. Through conditioning on the state at arrival and using PASTA and the above expression for the equilibrium probabilities, we obtain (cf. (4))

$$P(W > t) = \sum_{k=1}^{r} c_k \frac{y_k}{1-y_k} e^{-\mu(1-y_k)t}, \qquad t \ge 0.$$

Remark 5.2 The vector $(y_k, y_k^2, \ldots, y_k^r)$ is the row eigenvector of the rate matrix R for eigenvalue x_k, $k = 1,\ldots,r$.

5.4 The $E_r/M/1$ model

In this section we consider a single-server queue with exponential service times with mean $1/\mu$. The arrival process is not Poisson. The interarrival times are Erlang-r distributed with mean r/λ; i.e., the time between two arrivals is a sum of r independent exponential phases, each with mean $1/\lambda$. For stability we assume that the occupation rate

$$\rho = \frac{\lambda}{r} \cdot \frac{1}{\mu}$$

is less than one. The states of the Markov process describing this system are the pairs (i, j), where i denotes the number of jobs in the system and j the phase of the arrival process; i.e., $j = r$ means that already $r - 1$ phases of the interarrival have been completed, so there is only one phase to go before the next arrival. The transition-rate diagram is depicted in figure 5.

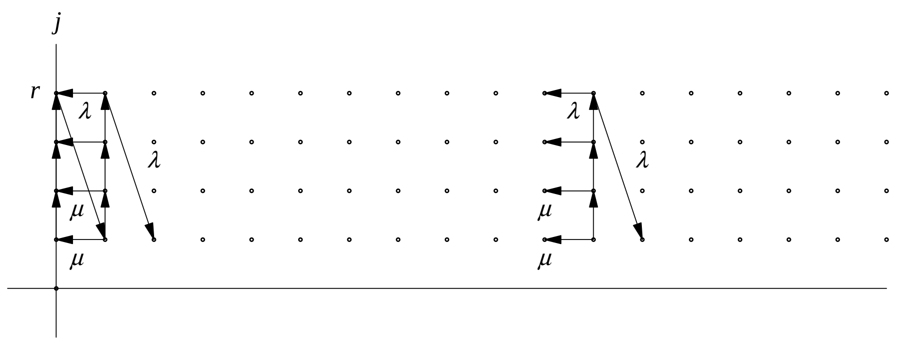

Figure 5: Transition-rate diagram for the $E_r/M/1$ model

Let us denote the state probabilities by $p(i, j)$. The equilibrium equations for the states (i, j) with $i \geq 1$ are formulated below.

$$\begin{aligned} p(i,1)(\lambda+\mu) &= p(i-1,r)\lambda + p(i+1,1)\mu, && (27)\\ p(i,j)(\lambda+\mu) &= p(i,j-1)\lambda + p(i+1,j)\mu, \qquad j = 2,\ldots,r. && (28) \end{aligned}$$

In these equations we now substitute

$$p(i,j) = y(j)x^i, \qquad i = 1,2,\ldots, j = 1,\ldots,r;$$

also for $i = 0$ and $j = r$ (so $p(0, r) = y(r)$). This leads to

$$\begin{aligned} y(j)x(\lambda+\mu) &= y(r)\lambda + y(1)x^2\mu,\\ y(j)(\lambda+\mu) &= y(j-1)\lambda + y(j)x\mu, \qquad j = 2,\ldots,r. \end{aligned}$$

Hence

$$\frac{y(j)}{y(j-1)} = \frac{\lambda}{\lambda+\mu-x\mu} = \text{constant} \equiv y,$$

so

$$y(j) = y^j, \qquad j = 1, \dots, r,$$

where y satisfies

$$\begin{aligned} x(\lambda + \mu) &= y^{r-1}\lambda + x^2\mu, \\ y(\lambda + \mu) &= \lambda + yx\mu. \end{aligned}$$

This gives that $x = y^r$ and

$$x = \left(\frac{\lambda}{\lambda + \mu - \mu x}\right)^r.$$

The above equation for x has exactly one root in $(0, 1)$, say x_1. Let $y_1 = \sqrt[r]{x_1}$. Then we eventually find

$$p(i, j) = c_1 y_1^j x_1^i, \qquad i = 1, 2, \dots, j = 1, \dots, r, \tag{29}$$

and this form is also valid for $p(0, r)$. The coefficient c_1 and the boundary probabilities $p(0, 1), \dots, p(0, r - 1)$ follow from the balance equations for the states $(0, 1), \dots, (0, r)$ and the normalization equation.

Solution (29) may also be written in matrix-geometric form; it is easily verified that

$$p_i = (p(i, 1), \dots, p(i, r)) = p_0 R^i, \qquad i = 0, 1, 2, \dots,$$

where

$$R = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ y_1 & y_1^2 & \cdots & y_1^r \end{pmatrix}.$$

In this model the excess probabilities of the waiting time are given by (cf. (4))

$$P(W > t) = x_1 e^{-\mu(1-x_1)t}, \qquad t \geq 0.$$

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