7 $G/M/1$ type models

In this chapter we consider $G/M/1$ type models, i.e., generalizations of the ordinary $G/M/1$ queue, and we state some of the main results; for a detailed exposition of the analysis of $G/M/1$ type models the reader is referred to [1].

In the $G/M/1$ queue customers arrive one by one with interarrival times identically and independently distributed according to an arbitrary distribution function $F_A(\cdot)$. The mean interarrival time is equal to $1/\lambda$. The service times are exponentially distributed with mean $1/\mu$. For stability we again require that the occupation rate $\rho = \lambda/\mu$ is less than one. The standard approach to determine the waiting time characteristics is through the Markov chain embedded on arrival instants. The state of this Markov chain can be described by i, where i is the number of customers in the system just before an arrival. To specify the transition probabilities of this Markov chain we first introduce the probabilities a_n defined as the probability that exactly n customers are served during an interarrival time (assuming there are at least n customers present at the start of the interarrival time). By conditioning on the length of the interarrival time it follows that

$$a_n = \int_{t=0}^{\infty} \frac{(\mu t)^n}{n!} e^{-\mu t} dF_A(t), \qquad n = 0, 1, 2, \ldots$$

Further let b_n denote the probability that *more than* n customers are served during an interarrival time, so

$$b_n = \sum_{k>n} a_k.$$

Then the transition probability matrix P takes the form

$$P = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & \cdots \\ b_1 & a_1 & a_0 & 0 & 0 & \cdots \\ b_2 & a_2 & a_1 & a_0 & 0 & \cdots \\ b_3 & a_3 & a_2 & a_1 & a_0 & \cdots \\ b_4 & a_4 & a_3 & a_2 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

The equilibrium probabilities p_i have a geometric form,

$$p_i = (1 - \sigma)\sigma^i, \qquad i = 0, 1, 2, \ldots,$$

where σ is the unique root in $(0, 1)$ of the equation

$$\sigma = E(e^{-\mu(1-\sigma)A});$$

the generic random variable A has distribution $F_A(\cdot)$.

In the following section we introduce a model, in continuous time, with the same transition strucure as the $G/M/1$ queue. But in this model the simple state i is replaced by a set of states (referred to as level i). Its equilbrium distribution will have a matrix-geometric form (or a sum of geometric terms).

7.1 Model

We consider a Markov process, the state space of which consists of the *boundary states* $(0, j)$ where j ranges from 0 to n, and a semi infinite strip of states (i, j) where i ranges from 1 to ∞ and j from 0 to m. The states are ordered lexicographically, that is, $(0, 0), (0, 1), \ldots, (0, n), (1, 0), \ldots, (1, m), (2, 0), \ldots, (2, m), \ldots$. The set of boundary states $\{(0, 0), (0, 1), \ldots, (0, m)\}$ will be called *level* 0, and the set of states $\{(i, 0), (i, 1), \ldots, (i, n)\}$, $i \geq 1$, will be called *level* i. We partition the state space according to these levels, and for this partitioning we assume that the generator Q is of the form

$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & \cdots \\ B_{10} & B_{11} & A_0 & 0 & 0 & \cdots \\ B_{20} & A_2 & A_1 & A_0 & 0 & \cdots \\ B_{30} & A_3 & A_2 & A_1 & A_0 & \cdots \\ B_{40} & A_4 & A_3 & A_2 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix},$$

where the matrix B_{00} is of dimension $(n+1) \times (n+1)$, $B_{0,1}$ of dimension $(n+1) \times (m+1)$, the matrices B_{i0}, $i \geq 1$, of dimension $(m+1) \times (n+1)$, and B_{11} and A_i, $i \geq 0$, are square matrices of dimension $m+1$. Let

$$A = \sum_{i=0}^{\infty} A_i.$$

Note that A is a generator; it describes the behavior of the Markov process Q in the (vertical) j-direction only. We assume that the Markov process Q is irreducible and that the generator A has exactly one communicating class. For the stability of Q we have the same result as theorem 6.3: the Markov process Q is ergodic if and only if

$$\pi A_0 e < \pi \sum_{i=2}^{\infty} (i-1) A_i e,$$

where e is the column vector of ones and $\pi = (\pi_0, \pi_1, \ldots, \pi_m)$ is the equilibrium distribution of the Markov process with generator A; so

$$\pi A = 0, \qquad \pi e = 1.$$

In the sequel we will assume that the Markov process Q is ergodic. Thus the equilibrium probabilities $p(i, j)$ exist. In the following section we describe the matrix-geometric results, which are very similar to the ones in section 6.2.

7.2 The matrix-geometric method

Provided the Markov process Q is ergodic, the equilibrium probability vectors p_i are given by the matrix-geometric form

$$p_i = (p(i, 0), p(i, 1), \ldots, p(i, m)) = p_1 R^{i-1}, \qquad i = 1, 2, \ldots, \tag{1}$$

where the matrix R is the *minimal nonnegative solution* of the matrix equation

$$\sum_{i=0}^{\infty} R^i A_i = 0. \qquad (2)$$

The matrix R has spectral radius less than one (so $I - R$ is invertable). Of course, from the matrix-geometric representation (1) it easily follows that R should satisfy (2); substitution of (1) into the equilibrium equations for the states at level i,

$$\sum_{k=0}^{\infty} p_{i-1+k} A_k = 0,$$

yields

$$p_{i-1} \sum_{k=0}^{\infty} R^k A_k = 0,$$

which, since $p_{i-1} > 0$, implies (2). The boundary equations for p_0 and p_1 are exactly the same as for the $M/M/1$ type model, treated in section 6.2. Hence, in comparison with the $M/M/1$ results, the only difference is that the matrix-quadratic equation for R is replaced by equation (2); this of course complicates the computation of R. Equation (2) can be rewritten as

$$R = -(A_0 + \sum_{k=2}^{\infty} R^k A_k) A_1^{-1}.$$

To solve this equation we first have to *truncate* the infinite sum at K say, and then compute an approximation for R by successive sustitutions, i.e.,

$$R_{l+1} = -(A_0 + \sum_{k=2}^{K} R_l^k A_k) A_1^{-1}, \qquad l = 0, 1, 2, \ldots$$

starting with $R_0 = 0$. The larger K, the better the resulting approximation for R, but also the higher the computational effort to compute this approximation.

We finally mention that the rate matrix R has the same probabilistic interpretation as described in section 6.2.

7.3 Spectral expansion method

Along the same lines as in section 6.3 it can be shown that the equilbrium probability vectors p_i can be expressed as

$$p_i = \sum_{j=0}^{m} c_j y_j x^{i-1}, \qquad i = 1, 2, \ldots$$

where $x_0, x_1, \ldots, x_m$ are the roots inside the unit circle of

$$\det(\sum_{k=0}^{\infty} x^k A_k) = 0. \tag{3}$$

The vector y_j, $j = 0, 1, \ldots, m$, is a nonnul solution of

$$y \sum_{k=0}^{\infty} x^k A_k = 0.$$

The difficulties, however, with this approach are (i) to prove that equation (3) has indeed $m+1$ (different) roots x with $|x| < 1$, and (ii) the computation of these roots. In the next chapter we will consider a special class of $G/M/1$ models, for which these difficulties can be resolved.

References

[1] M.F. Neuts, *Matrix-geometric solutions in stochastic models*. The John Hopkins University Press, Baltimore, 1981.