10 Queueing systems with general interarrival times

In this chapter we shortly discuss single-station queueing systems with generally distributed interarrival times. This is relevant in situations where, for example, the arrivals are generated by the output of the preceding machine. Because the arrival process in these models is not Poisson, the PASTA property does not hold and hence also the mean value approach does not work. First, we will discuss the $G/M/1$ system for which an exact analysis is still possible. After that we present some approximative results for the $G/G/1$ system

10.1 The $G/M/1$ system

For the $G/M/1$ system exact analysis is possible. It can be shown that, on arrival instants of jobs, the number of jobs in the system is geometrically distributed. More specifically, let the random variable A be the interarrival time, with density $f_A(\cdot)$, and let μ be the parameter of the exponential processing time distribution. Then there is a unique solution σ between 0 and 1 of the equation

$$x = E(e^{-\mu(1-x)A}). \tag{1}$$

The probability, a_k, that a job finds on arrival k other jobs in the system is now given by

$$a_k = (1-\sigma)\sigma^k, \qquad k = 0, 1, \ldots.$$

See, e.g., [3] for a proof of the above results. Similarly as in section 4.4, we can now show that the throughput time in the $G/M/1$ queue is exponentially distributed with parameter $\mu(1-\sigma)$, so

$$P(S \leq t) = 1 - e^{-\mu(1-\sigma)t}, \qquad t \geq 0.$$

Note that the parameter σ depends through (1) on the complete distribution of the random variable A. Hence, also the mean throughput time depends on this complete distribution (and not only on the first two moments of A).

Example 10.1 *($M/M/1$)*
For exponentially distributed interarrival times with parameter λ we have $f_A(t) = \lambda e^{-\lambda t}$ and hence

$$E(e^{-\mu(1-x)A}) = \int_0^\infty \lambda e^{-\lambda t} e^{-\mu(1-x)t} dt = \frac{\lambda}{\lambda + \mu(1-x)}.$$

Hence, equation (1) reduces to

$$x = \frac{\lambda}{\lambda + \mu(1-x)},$$

so

$$x(\lambda + \mu - \mu x) - \lambda = (x-1)(\lambda - \mu x) = 0.$$

Thus the desired solution between 0 and 1 is given by $\sigma = \lambda/\mu = \rho$ and the distribution on arrival instants is given by

$$a_k = (1-\rho)\rho^k, \quad k = 0, 1, 2, \ldots$$

Note that this distribution is exactly the same as the equilibrium distribution of the $M/M/1$. This is of course no surprise, because here we have Poisson arrivals and so the PASTA property holds.

Example 10.2 *($E_2/M/1$)*
Suppose that the interarrival times are Erlang-2 distributed with mean 2/3, so

$$E(e^{-sA}) = \left(\frac{3}{3+s}\right)^2.$$

Further assume that $\mu = 4$ (so $\rho = 3/2 \cdot 1/4 = 3/8 < 1$). Then equation (1) reduces to

$$x = \left(\frac{3}{7-4x}\right)^2.$$

Thus

$$x(7-4x)^2 - 9 = (x-1)(4x-9)(4x-1) = 0.$$

Hence the desired solution between 0 and 1 is given by $\sigma = 1/4$ and

$$a_k = \frac{3}{4}\left(\frac{1}{4}\right)^k, \quad k = 0, 1, 2, \ldots$$

10.2 The $G/G/1$ system

For the $G/G/1$ system with general interarrival times and arbitrary processing times we present some approximations. The simplest approximation for the mean waiting time assumes that the randomness of the interarrival times has more or less the same effect on the mean waiting time as the randomness in the service times. Denoting the coefficient of variation of the interarrival times by c_A and the coefficient of variation of the processing times by c_B, the approximation is given by (see, e.g., [4, 5])

$$E(W) \approx \frac{\rho}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2} \cdot E(B). \tag{2}$$

Other approximations that are proposed are (see, e.g., [1])

$$E(W) \approx \frac{\rho}{1-\rho} \cdot \frac{(1+c_B^2)\,(c_A^2 + \rho^2 c_B^2)}{2\,(1+\rho^2 c_B^2)} \cdot E(B) \tag{3}$$

and

$$E(W) \approx \frac{\rho}{1-\rho} \cdot \frac{(1+c_B^2)\,((2-\rho)c_A^2 + \rho c_B^2)}{2\,(2-\rho+\rho c_B^2)} \cdot E(B) \tag{4}$$

These approximations depend only on the first two moments of the interarrival and service times. Note that for ρ close to 1, the above approximations are (nearly) the same. Also they are exact for the case of Poisson arrivals (for which $c_A = 1$). As a rule of thumb, these

	c_A^2		
	1/4	1/3	1/2
app. (2)	2.250	2.625	3.375
app. (3)	2.117	2.507	3.287
app. (4)	2.122	2.512	3.290
exact	2.076	2.466	3.250

Table 1: Comparison of the approximations of the mean waiting times with exact results, for $\rho = 0.9$, $E(B) = 1$, $c_B^2 = 1/4$ and various values of c_A^2

approximations work well as long as $c_A^2 \leq 2$. In table 10.2 we compare the approximations with exact results in case of Erlang distributed interarrival and processing times. We have chosen $\rho = 0.9$, $E(B) = 1$, $c_B^2 = 1/4$ and $c_A^2 = 1/4$, 1/3 and 1/2, respectively.

Under heavy load conditions (ρ close to 1) the waiting time distribution in the $G/G/1$ system is approximately exponentially distributed with mean given by (2) (or (3) or (4)); see, e.g., [2].

By Little's law, we can also obtain an approximation for the mean number in the system $E(L)$. Let us now consider a (rough) approximation for the distribution of the number of jobs in the system, p_k, $k = 0, 1, \ldots$ The fraction of time the machine is idle is equal to $1 - \rho$. Hence, $p_0 = 1 - \rho$. Now let us assume that the remaining probabilities have a geometric form, i.e., $p_k = a\sigma^{k-1}$ for $k = 1, 2, \ldots$. Since

$$1 = \sum_{k=0}^{\infty} p_k = 1 - \rho + \frac{a}{1 - \sigma},$$

we get

$$a = \rho(1 - \sigma).$$

Further,

$$E(L) = \sum_{k=0}^{\infty} k p_k = \frac{a}{(1 - \sigma)^2}.$$

These two equations may be solved for a and σ. This finally yields the following approximation for the probabilities p_k,

$$p_k = \begin{cases} 1 - \rho, & k = 0, \\ \rho(1 - \sigma)\sigma^{k-1}, & k = 1, 2, \ldots, \end{cases}$$

where $\sigma = (E(L) - \rho)/E(L)$.

Example 10.3 In a workcell consisting of a single robot raw material is delivered at a rate of one lot every 8 hours. The standard deviation of the delivery time is 4 hours (as past records indicate). The average cycle time for a lot is 6 hours with a standard deviation of 2 hours. According to (2) the prediction for the production lead time of a lot is roughly

9.25 hours. What would be the reduction in the lead time if the material supply could be made more reliable, for instance with only one hour standard deviation? In this case the lead time is reduced to approximately 7 hours, which is an improvement of almost 25%. Hence, simple formulas like (2) may be used to quickly determine rough-cut answers for the mean waiting time.

10.3 Departure process of the $G/G/1$ system

As we will see later, the $G/G/1$ system forms the building block in the development of approximation techniques for the analysis of production networks. In a network the departures from one machine are arrivals to another machine. Hence, it is useful to be able to characterize the departure process of the $G/G/1$ system. In general, the interdeparture times are not independent, but as an approximation, we will act as if. By conservation of flow, the departure rate is equal to the arrival rate, so the mean of the interdeparture time equals the mean interarrival time. A simple approximation for the squared coefficient of variation of the interdeparture time, c_D^2, is given by (see, e.g., [4])

$$c_D^2 \approx (1-\rho^2)c_A^2 + \rho^2 c_B^2.$$

This approximation is intuitively appealing: under light load conditions (ρ close to 0), c_D^2 is approximately equal to c_A^2 and under heavy load conditions (ρ close to 1) it is approximately equal to c_B^2.

References

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