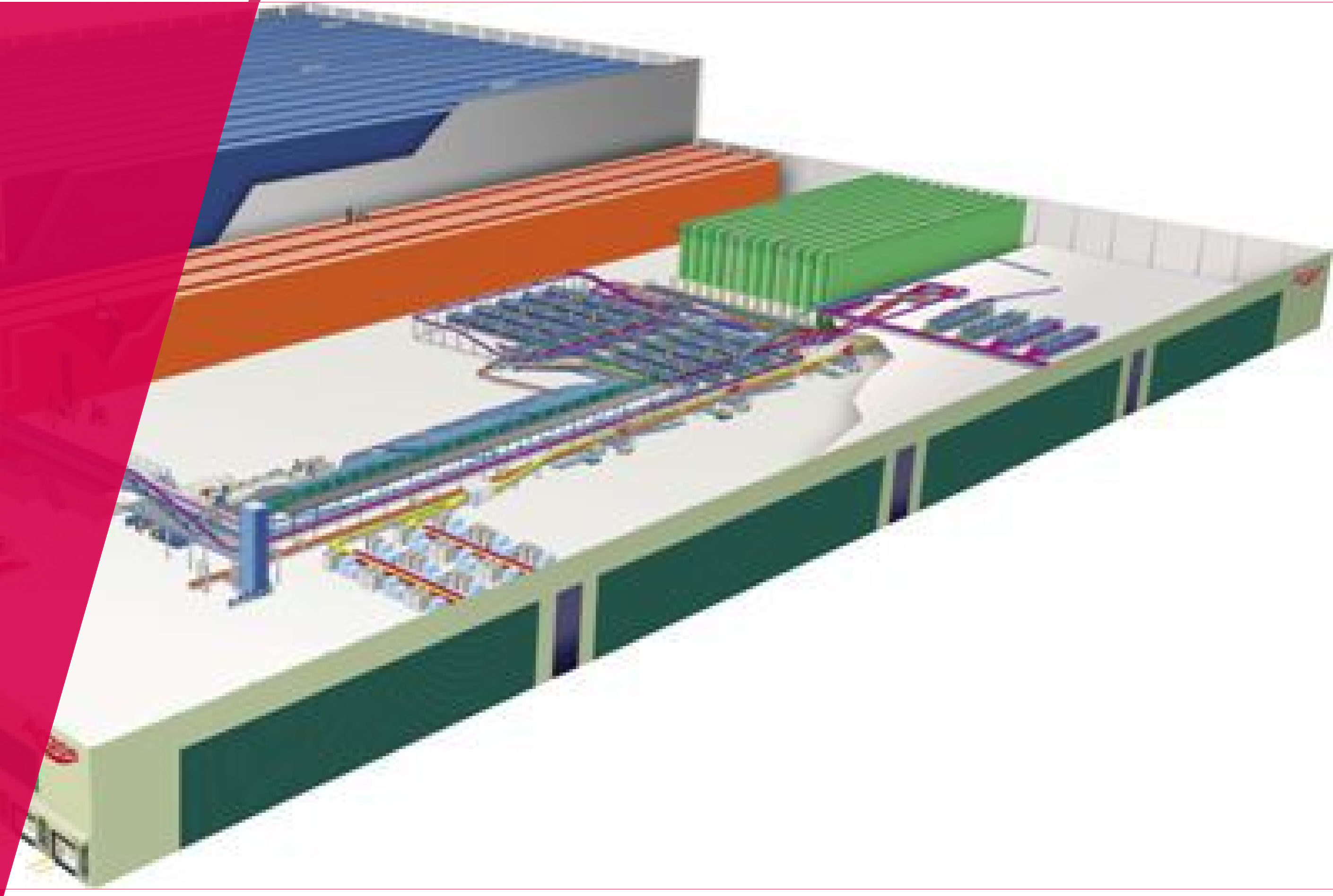
Technische Universiteit
Eindhoven
University of Technology

Facility Logistics Management

Ivo Adan



Discrete distributions: Poisson

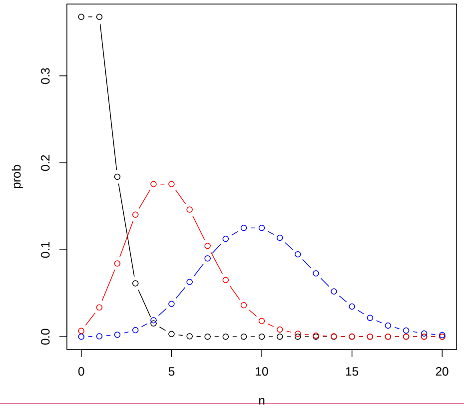
- Poisson random variable X with parameter $\lambda > 0$

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

- Expectation and variance

$$\mathbb{E}[X] = \lambda, \quad \mathrm{Var}[X] = \lambda$$

- Probability plot for $\lambda = 1, 5, 10$



Discrete distributions: Geometric

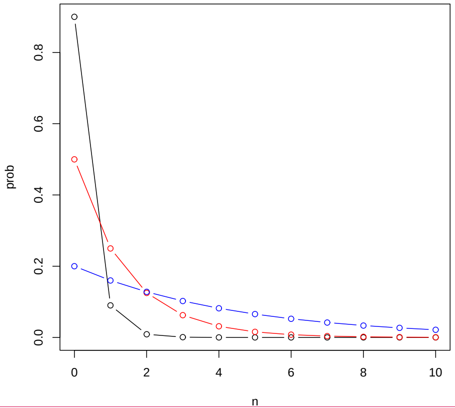
- Geometric random variable X is the number of trials till the first success (with probability p)

$$\mathrm{P}(X = k) = (1-p)^{k-1}p, \quad k = 1, 2, \ldots$$

- Expectation and variance

$$\mathbb{E}[X] = \frac{1}{p}, \quad \mathrm{Var}[X] = \frac{1-p}{p}$$

- Probability plot for success probability $p = 0.9, 0.5, 0.2$



Continuous distributions: Exponential

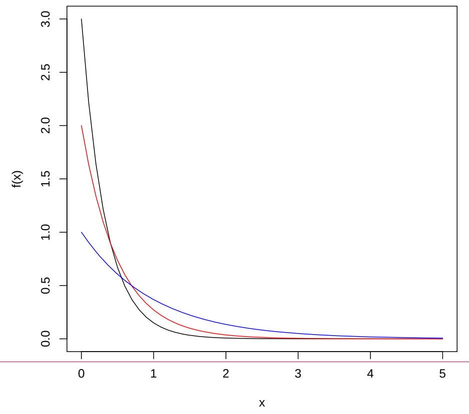
- Exponential random variable X with parameter (or rate) $\lambda > 0$ has density

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

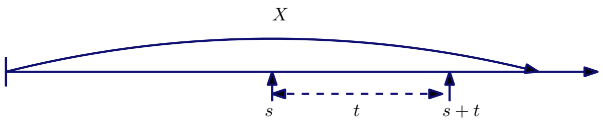
- Distribution, expectation and variance

$$P(X \le x) = 1 - e^{-\lambda x}, \quad \mathbb{E}[X] = \frac{1}{\lambda}, \quad \mathrm{Var}[X] = \frac{1}{\lambda^2}$$

- Probability density plot for $\lambda = 1, 2, 3$



Memoryless property of Exponential



- For all $t, s > 0$

$$P(X > t + s | X > s) = P(X > t)$$

- This follows from

$$\begin{aligned} P(X > t + s | X > s) &= \frac{P(X > t + s, X > s)}{P(X > s)} \\ &= \frac{P(X > t + s)}{P(X > s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} \end{aligned}$$

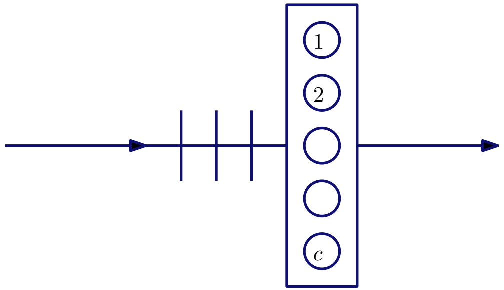
- In words: “Overshoot” of X at time s is again Exponential with parameter λ so used is as good as new!
- Alternative form for small $\Delta > 0$

$$\mathrm{P}(X \leq s + \Delta | X > s) = 1 - e^{-\lambda\Delta} \approx \lambda\Delta$$

Variability interactions - Queueing

- Arrival process:
  - Arrivals can consist of single jobs or batches, there may be single or multiple flows
- Service (or production) process:
  - Station can have a single server, or multiple (non-) identical ones
  - Service discipline can be:
    - First-come first-served (FCFS)
    - Last-come first-served (LCFS)
    - Shortest process time first (SPTF)
    - Random
- Queue (or buffer):
  - There may be ample (unlimited) queue space, or limited (or even no) queue space

Variability interactions - Queueing



- $A/B/c$ notation:
  - A specifies distribution of inter-arrival times
  - B distribution of the service times
  - c number of servers
- Implicit assumption: Service is FCFS and unlimited queueing space
- Possible letters for distribution: M (Exponential), U (Uniform), D (Deterministic), G (General)
- Examples: $D/D/1$, $M/M/1$, $M/G/1$, $G/G/c$
- Extensions: $M/M/1/b$ (Limited queueing space of b places)

Poisson arrival flow

- Times between arrivals are independent and Exponential with rate λ
- Memoryless property

$$\mathrm{P}(\text{arrival in } (t, t+\Delta)) = 1 - e^{-\lambda\Delta} \approx \lambda\Delta$$

So "truly unpredictable arrivals": in each small interval Δ there is an arrival with probability $\lambda\Delta$!

- Number of arrivals in interval $(0, t)$ is Poisson distributed with parameter λt

$$\mathrm{P}(k \text{ arrivals in } (0, t)) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \ldots$$

- Clustered arrivals (since inter-arrival time density $f(x) = \lambda e^{\lambda x}$ is maximal for $x = 0$)

- Merging of two Poisson flows with rates λ_1 and λ_2 is again Poisson with rate $\lambda_1 + \lambda_2$, since

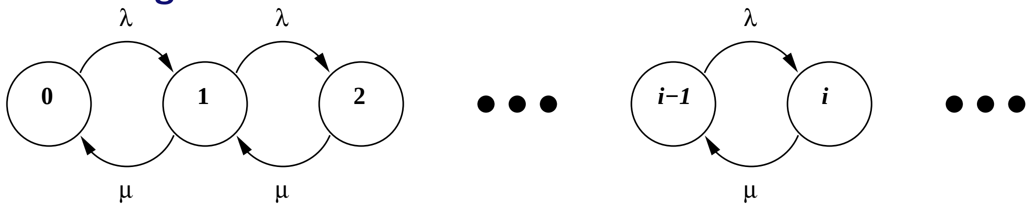
$$\mathrm{P}(\text{arrival in } (t, t+\Delta)) \approx \lambda_1\Delta + \lambda_2\Delta = (\lambda_1 + \lambda_2)\Delta$$

- Random splitting of Poisson flow with rate λ and splitting probability p is again Poisson with rate $p\lambda$, since

$$\mathrm{P}(\text{arrival in } (t, t+\Delta)) \approx \lambda\Delta p = p\lambda\Delta$$

Single Exponential server

- Poisson arrivals with rate λ and single server with Exponential service times with rate μ
- Stability: $\rho = \lambda/\mu < 1$
- p_i is long-run probability (or fraction of time) of finding i jobs in the system
- Flow diagram



- Balance equations: Flow from state i to $i-1$ = Flow from state $i-1$ to i
- Balance equations: $p_i\mu = p_{i-1}\lambda$

$$p_i = p_{i-1}\frac{\lambda}{\mu} = p_{i-2}\left(\frac{\lambda}{\mu}\right)^2 = \cdots = p_0\left(\frac{\lambda}{\mu}\right)^i = p_0\rho^i$$

- p_0 is probability that server is idle: $p_0 = 1 - \frac{\lambda}{\mu} = 1 - \rho$