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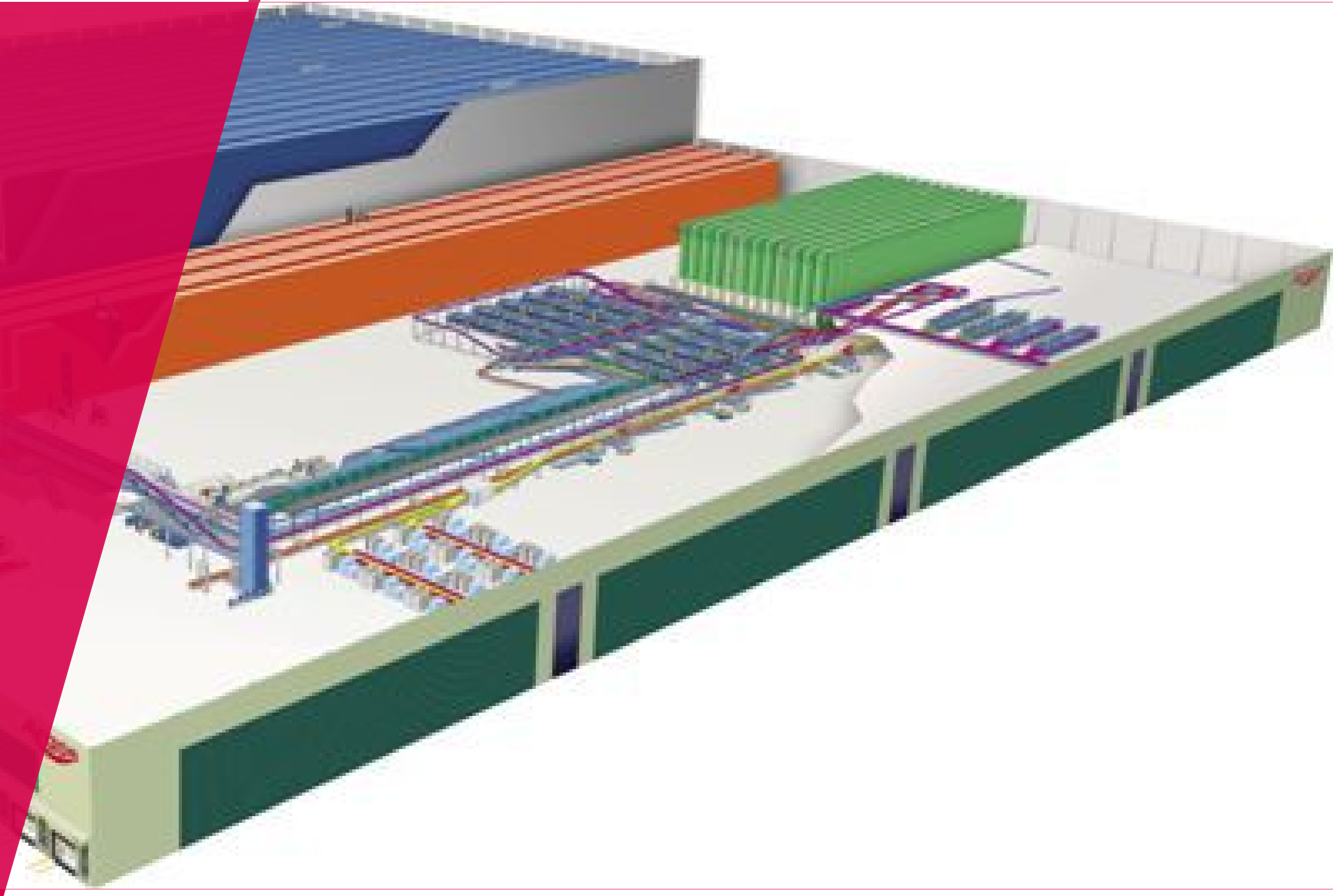

Facility Logistics Management

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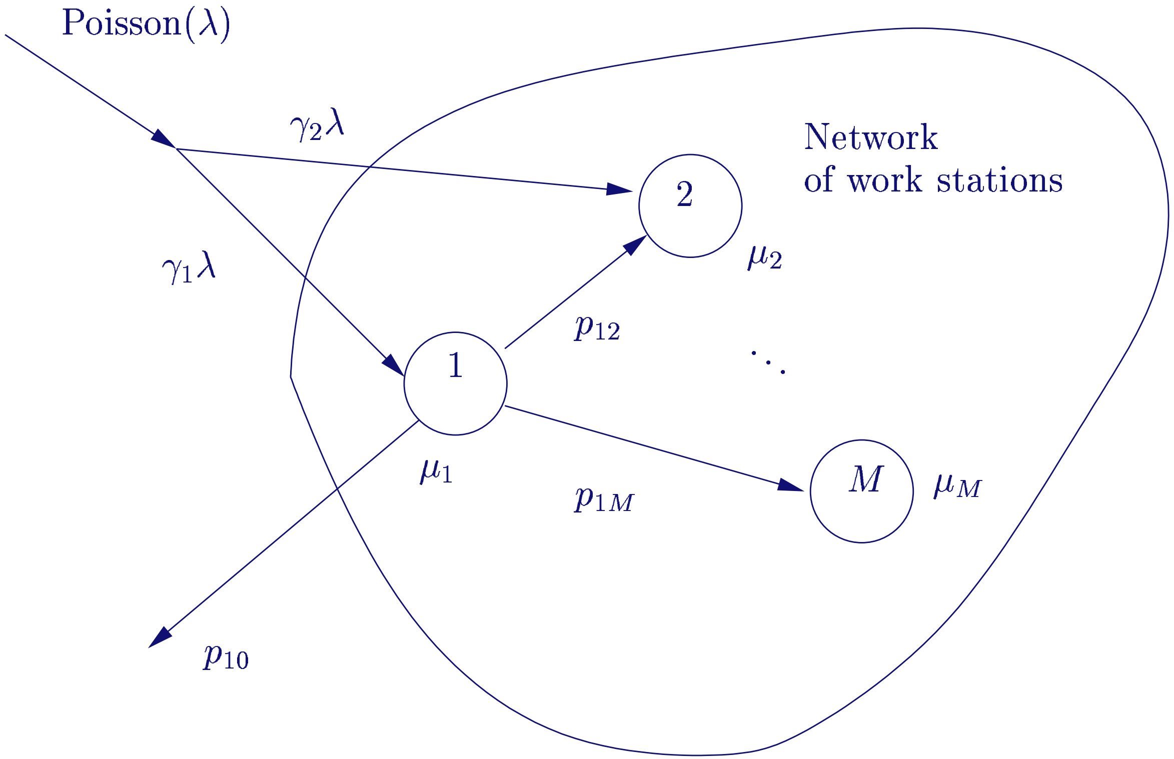
Exponential Queueing Networks: Open Jackson Networks

- Stations $1, \dots, M$
- Station m has c_m parallel identical servers
- Jobs arrive according to Poisson stream with rate λ
- Arriving job joins station m with probability γ_m
- Service times in station m are Exponential with rate μ_m
- Service is in order of arrival
- Routing: Job moves from station m to n with probability p_{mn} and leaves system with probability p_{m0}

Exponential Queueing Networks: Open Jackson Networks



Exponential Queueing Networks: Open Jackson Networks

- λ_m is total arrival rate to station m

$$\lambda_m = \gamma_m \lambda + \sum_{n=1}^{M} \lambda_n p_{nm}, \quad m = 1, \dots, M$$

- Stability

$$\rho_m = \frac{\lambda_m}{c_m \mu_m} < 1 \quad m = 1, \dots, M$$

- Bottleneck station b

$$\rho_b = \max_{1 \le m \le M} \rho_m$$

Exponential Single Server Network: Open Jackson Networks

- Network states $\underline{k} = (k_1, \dots, k_M)$ where k_m is number of jobs in station m
- State probabilities $p(k_1, k_2, \dots, k_M)$ satisfy balance equations $(c_m = 1)$

$$p(\underline{k})\left(\lambda + \sum_{m=1}^{M} \mu_m \epsilon(k_m)\right) = \sum_{m=1}^{M} p(\underline{k} + \underline{e}_m)\mu_m p_{m0} + \sum_{n=1}^{M}\sum_{m=1}^{M} p(\underline{k} + \underline{e}_n - \underline{e}_m)\mu_n p_{nm}\epsilon(k_m) + \sum_{m=1}^{M} p(\underline{k} - \underline{e}_m)\lambda\gamma_m\epsilon(k_m)$$

where $\underline{e}_m = (0, \dots, 1, \dots, 0)$ with 1 at place m and $\epsilon(k) = \begin{cases} 1 & \text{if } k > 0 \\ 0 & \text{else} \end{cases}$

- State probabilities $p(k_1, k_2, \dots, k_M)$ have product form

$$p(\underline{k}) = p_1(k_1)p_2(k_2)\cdots p_M(k_M)$$

where

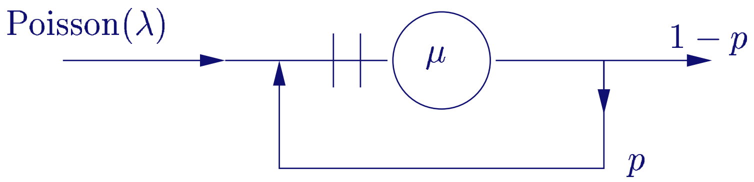
$$p_m(k_m) = (1 - \rho_m)\rho_m^{k_m} \quad k_m = 0, 1, \dots$$

and $\rho_m = \lambda_m / \mu_m$

- Jackson's miracle: This is just the product of $M/M/1$ solutions!

Exponential Single Server Network: Open Jackson Networks

- Queue lengths at stations are independent (when you take a snapshot)!
- Inflow to station i is in general not Poisson! But you can do as if
- Example:



$\mu = 1/\epsilon$, $p = 1-\epsilon$ (so $\mu(1-p) = 1$), $\lambda \ll 1$

- Arrival pattern

- So not Poisson!
- Question: What is the solution for an exponential multi-server network?
- Answer: This is the product of $M/M/c_m$ solutions!

Exponential open job shops: Open Jackson Networks

- Queue lengths at workstations are independent (if you take a snapshot)!
- Inflow to workstation m is in general not Poisson! But you can "do as if"
- Infinite server station $c_m = \infty$

$$p_m(k_m) = e^{-\rho_m} \frac{\rho_m^{k_m}}{k_m!} \quad \text{(Poisson distribution)}$$

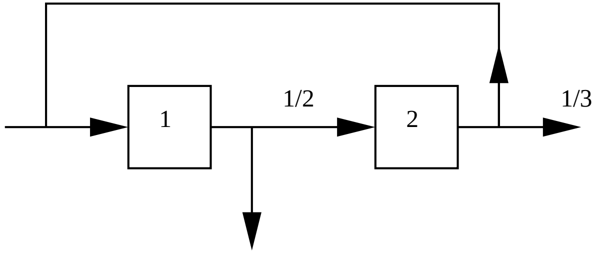
where $\rho_m = \lambda_m / \mu_m$

- Distribution for $c_m = \infty$ also valid for general service time distribution!
- Infinite server stations useful to describe transportation delay
- Product form result also valid for fixed route $C_1, C_2, \dots, C_n$ in which case

$$\lambda_m = \lambda \sum_{i=1}^{n} \mathbf{1}[C_i = m]$$

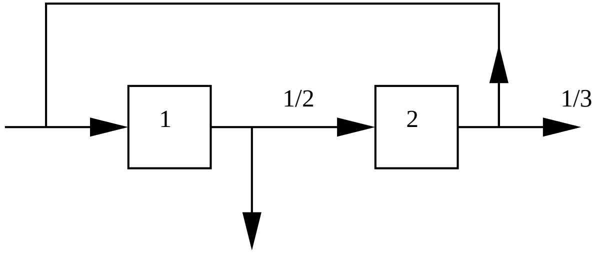
where $\mathbf{1}[C_i = m] = 1$ if $C_i = m$ and 0 otherwise

Exponential open job shops: Example



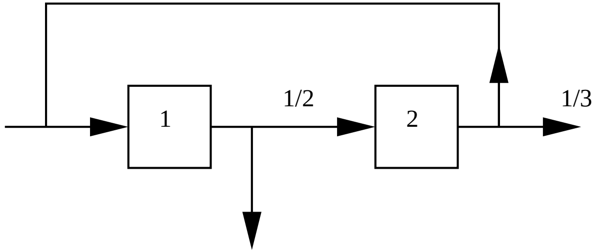
- $M = 2$ workstations
- $c_1 = c_2 = 1$ (single server network)
- Arriving jobs join workstation 1 ($\gamma_1 = 1$)
- Routing: $p_{12} = 1 - p_{10} = \frac{1}{2}$, $p_{21} = 1 - p_{20} = \frac{2}{3}$
- Question: What is mean time spent in system?
- Answer: $\lambda_1 = \frac{3}{2}\lambda$, $\lambda_2 = \frac{3}{4}\lambda$ (note: $\frac{\lambda_1}{\lambda} = \frac{3}{2}$ is mean number of visits of a job to station 1)
- Stability: $\lambda_1 < \mu_1$, $\lambda_2 < \mu_2$, so $\lambda < \min\{\frac{2}{3}\mu_1, \frac{4}{3}\mu_2\}$
- $E(S_1) = \frac{1}{\mu_1 - \lambda_1}$, $E(S_2) = \frac{1}{\mu_2 - \lambda_2}$, $E(S) = \frac{3}{2}E(S_1) + \frac{3}{4}E(S_2)$

Exponential open job shops: Example



- $M = 2$ workstations
- $c_1 = c_2 = 1$ (single server network)
- Arriving jobs join workstation 1 ($\gamma_1 = 1$)
- Routing: $p_{12} = 1 - p_{10} = \frac{1}{2}$, $p_{21} = 1 - p_{20} = \frac{2}{3}$
- Question: What is mean time spent in system in case of fixed routing: 1, 2, 1?
- Answer: $\lambda_1 = 2\lambda$, $\lambda_2 = \lambda$
- Stability: $\lambda_1 < \mu_1$, $\lambda_2 < \mu_2$, so $\lambda < \min\{\frac{1}{2}\mu_1, \mu_2\}$
- $E(S_1) = \frac{1}{\mu_1 - \lambda_1}$, $E(S_2) = \frac{1}{\mu_2 - \lambda_2}$, $E(S) = 2E(S_1) + E(S_2)$

Exponential open job shops: Example



- $M = 2$ workstations
- $c_1 = c_2 = 1$ (single server network)
- Arriving jobs join workstation 1 ($\gamma_1 = 1$)
- Routing: $p_{12} = 1 - p_{10} = \frac{1}{2}$, $p_{21} = 1 - p_{20} = \frac{2}{3}$
- Question: What is mean time spent in system in case of transportation delays T between station 1 and 2?
- Answer: Mean number of times a job transported from station 1 to 2 is $\frac{3}{4}$, so add $\frac{3}{4}T$ to mean flow time $\mathrm{E}(S)$

General open job shops

- Stations $1, \dots, M$
- Station m has c_m parallel identical servers
- Jobs arrive according to general stream with general inter-arrival times A, with with rate $\lambda = 1/\mathrm{E}(A)$
- Arriving job joins station m with probability γ_m
- Service times in station m are General with mean $E(B_m)$ and coefficient of variation c_{B_m}
- Service is in order of arrival
- Routing: Job moves from station m to n with probability p_{mn} and leaves system with probability p_{m0}

General open job shops

- Lesson from Jackson networks: Each station can be analyzed in isolation with appropriate arrival process
- Approach for "large randomly routed networks":
  - Arrival process at each station is "random", thus approximately Poisson with rate λ_m

$$\lambda_m = \gamma_m \lambda + \sum_{n=1}^{M} \lambda_j p_{nm}, \quad m = 1, \dots, M$$

  - Model each workstation m as $M/G/c$ with Poisson arrival rate λ_m, $c = c_m$ and $B = B_m$

$$\mathrm{E}(S_m) \approx \frac{1}{2}(1 + c_{B_m}^2) \frac{\Pi_{W_m}}{1 - \rho_m} \frac{\mathrm{E}(B_m)}{c_m} + \mathrm{E}(B_m)$$

where Π_{W_m} is probability of waiting in corresponding $M/M/c$ with $\lambda = \lambda_m$, $\mu_m = 1/\mathrm{E}(B_m)$ and $c = c_m$