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Facility Logistics Management

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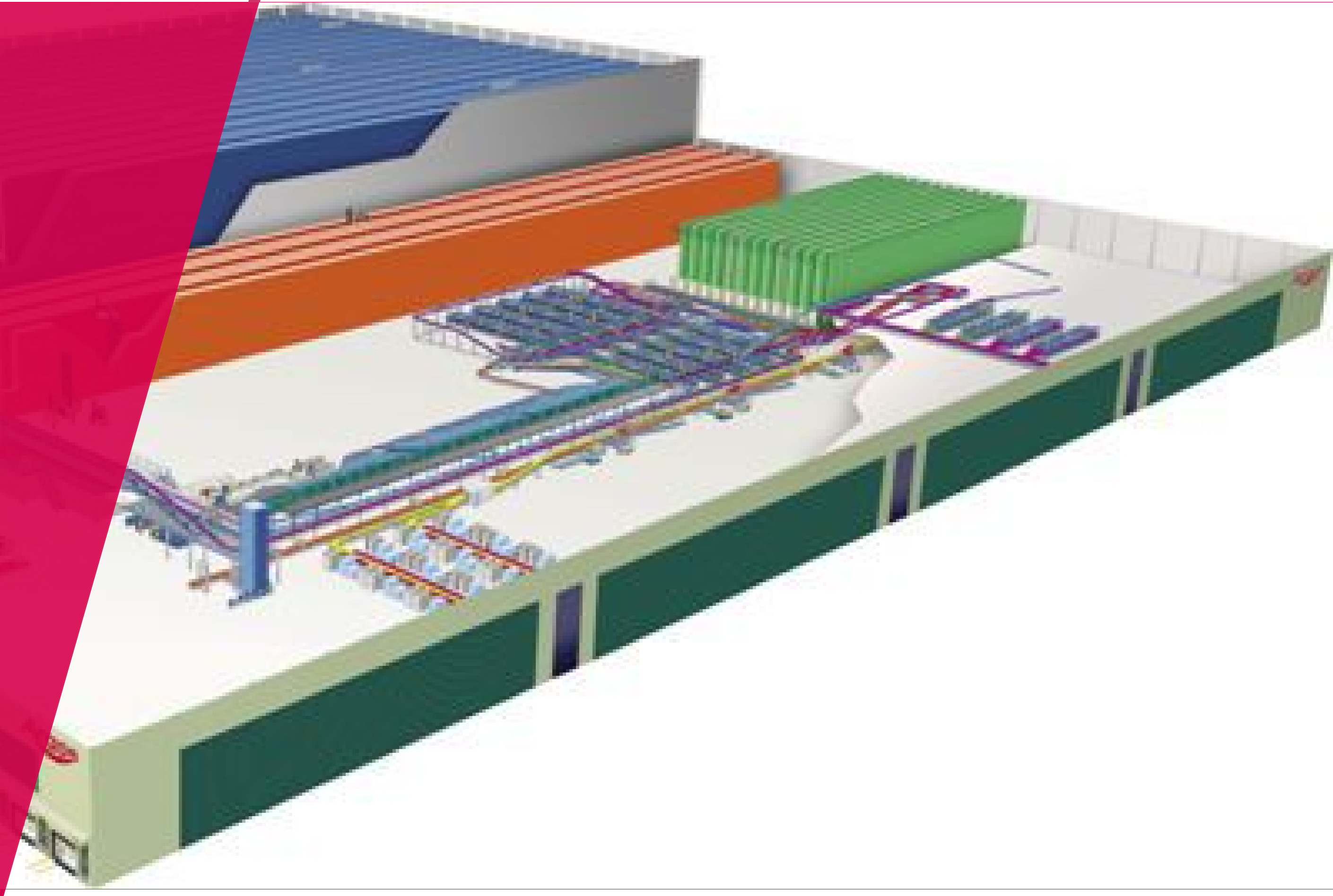
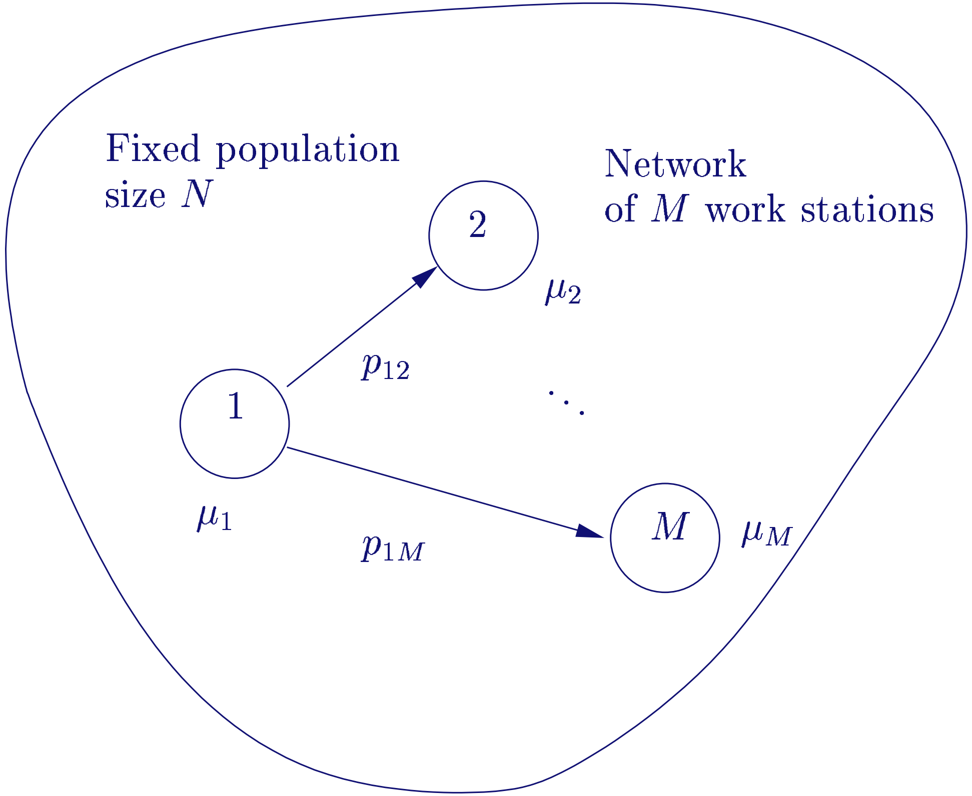


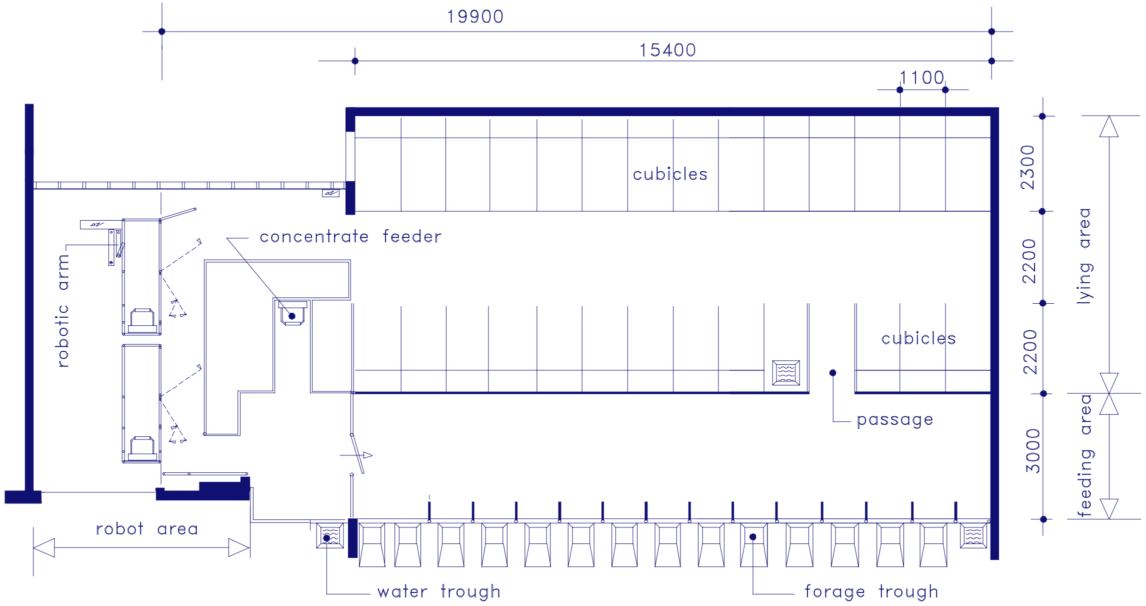
Exponential closed networks: Closed Jackson Networks

- Workstations $1, \dots, M$
- Workstation m has c_m parallel identical machines
- N circulating jobs (N is the population size)
- Service times in workstation m are Exponential with rate μ_m
- Service is in order of arrival
- Routing: job moves from workstation m to n with probability p_{mn}

Exponential closed networks: Closed Jackson Networks



Example: Robotic barn



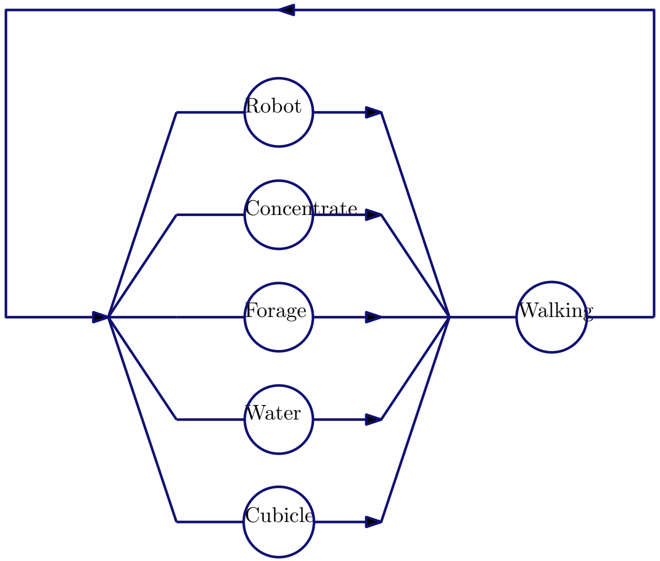
- Design issues
  - Number of robots
  - Number and location of feeders, troughs, drinking places, cubicles

Example: Robotic barn

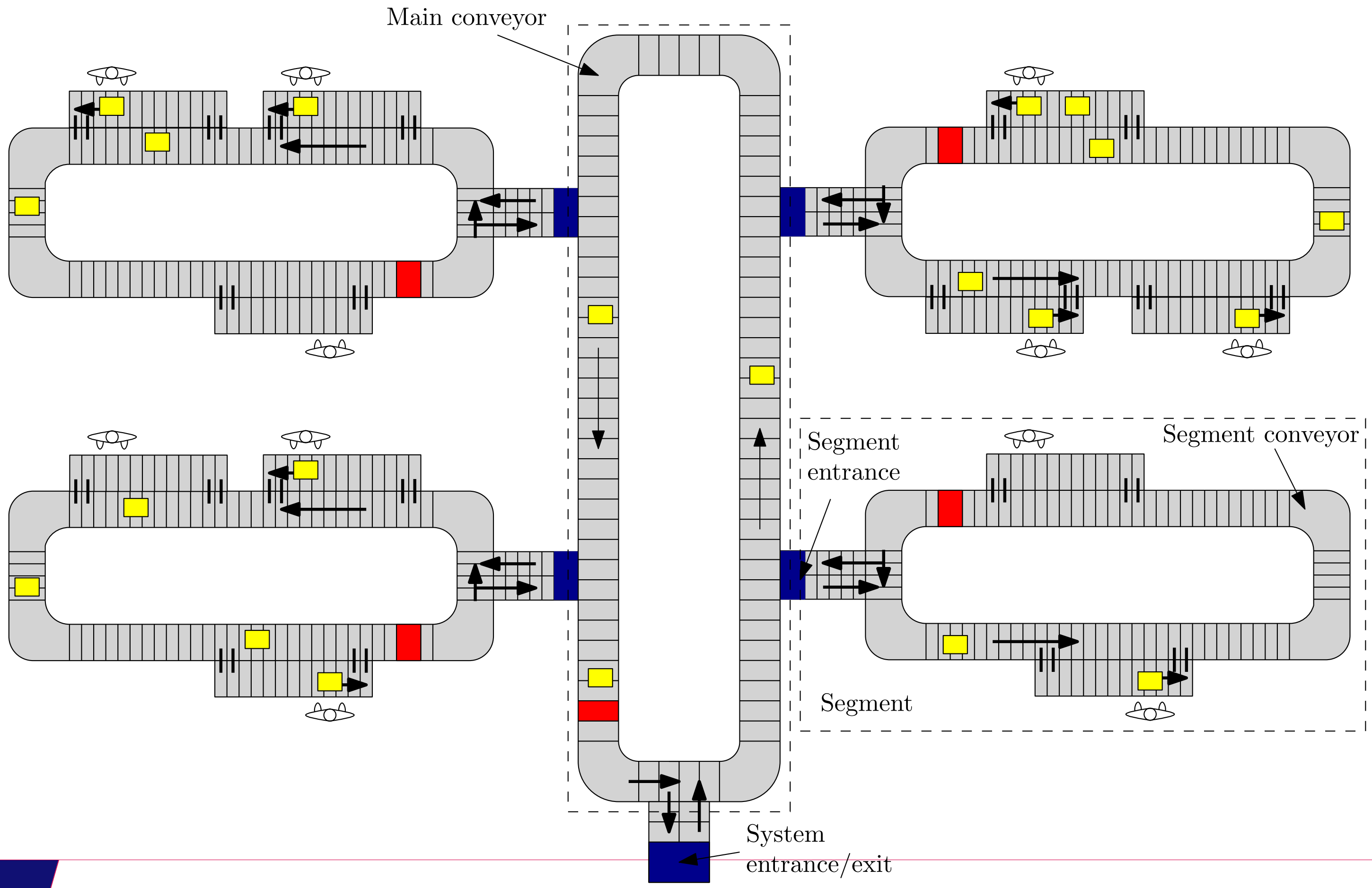
- Closed network with N circulating cows and workstations:

  1. Milking robot
  2. Concentrate feeder
  3. Forage lane
  4. Water trough
  5. Cubicle
  6. Walking

Example: Robotic barn



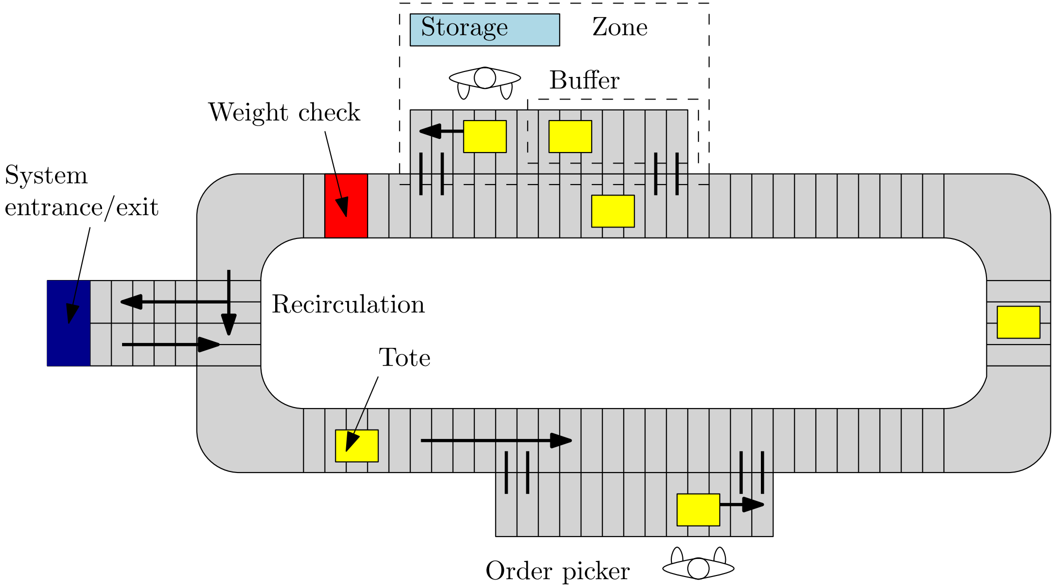
Example: Zone-Picking



Example: Zone-Picking

- Key features
  - Man-to-goods solution
  - Work-load control at multiple levels
  - Limited buffer space at zones
  - Recirculation (on main and local loops)
- Design issues
  - Layout of the network
  - Number and size of zones
  - Location of items
  - Number of pickers
  - Required WIP level

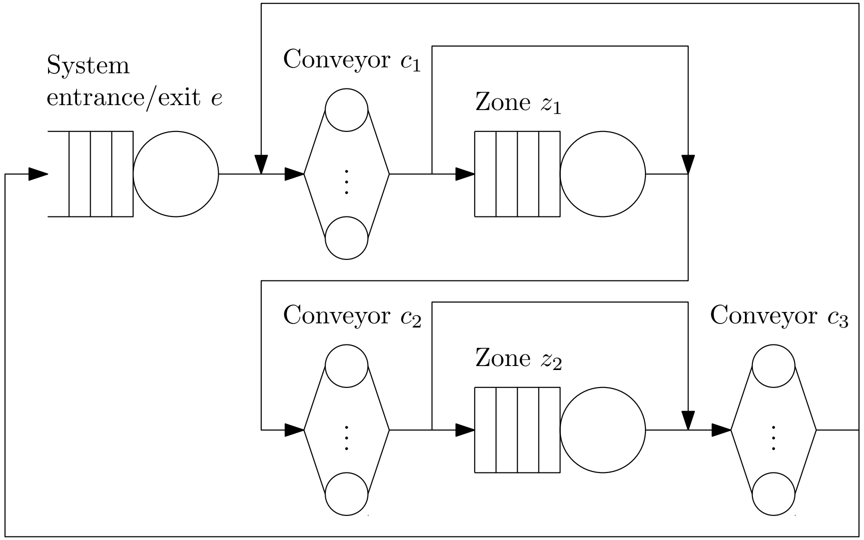
Example: Single Zone



Example: Single Zone

- Closed network with N circulating totes and workstations:
  1. System entrance/exit
  2. Conveyors
  3. Picking zones

Example: Single Zone



Example: KIVA system

Kiva robot

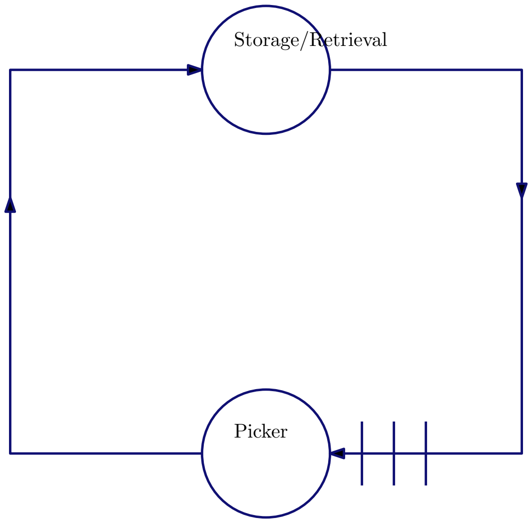
Operating in warehouse

- Key features
  - High throughput capability, flexibility and scalability
- Design issues
  - Number and routing of robots
  - Storage location of racks
  - Number and location of pick locations

Example: KIVA system

- Closed queueing network model with N circulating robots and workstations:
  1. Picking station
  2. Storage/retrieval

Example: KIVA system

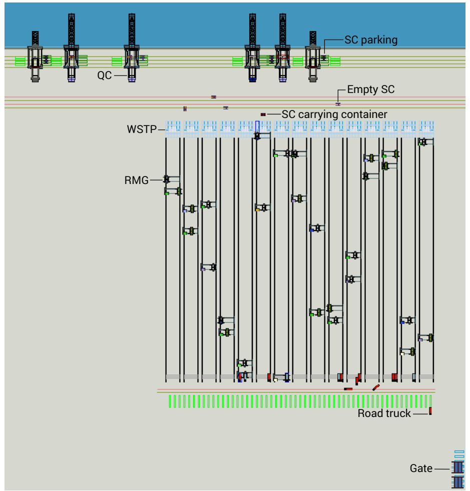


Example: Container terminal

Straddle carrier

Container terminal

Schematic layout



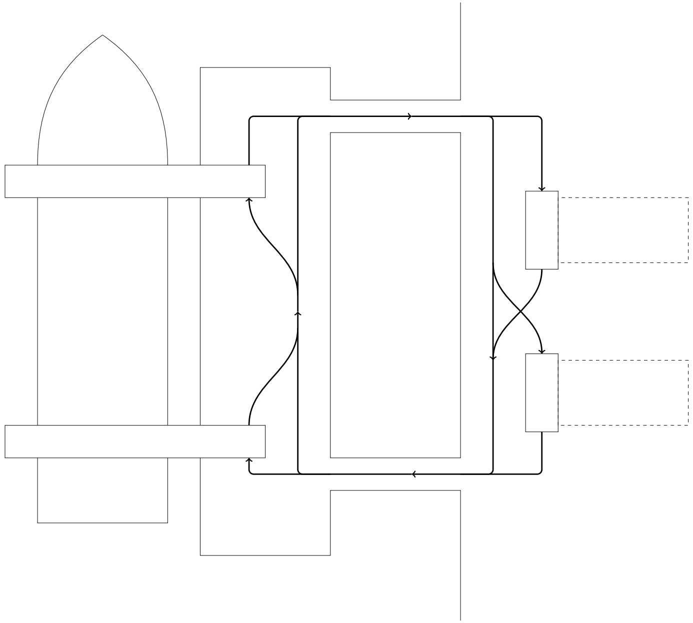
- Design issues
  - Number of quay cranes
  - Number of straddle carriers

Example: Container terminal

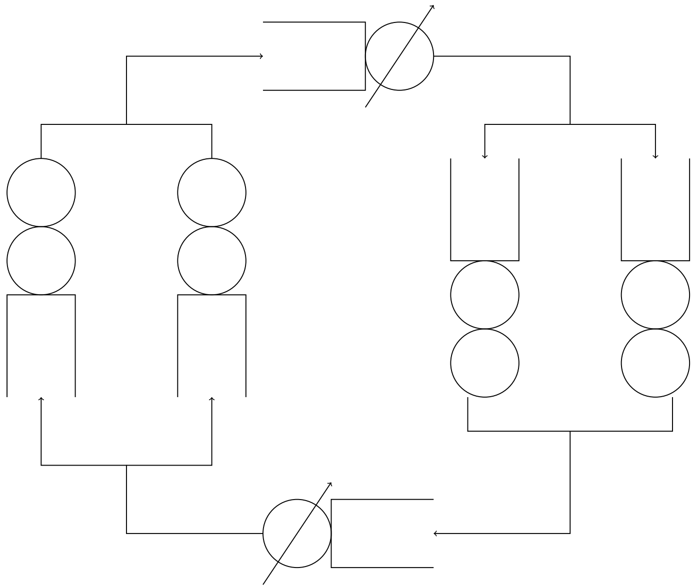
- Closed queueing network model with N circulating straddle carriers and workstations:

  1. Quay cranes
  2. Gantry cranes
  3. Transportation

Example: Container terminal



Example: Container terminal



Exponential closed Single server network: Closed Jackson Networks

- Network states $\underline{k} = (k_1, \ldots, k_M)$ where k_m is number in station m: Note $\sum_{m=1}^{M} k_m = N$ so $\binom{N+M-1}{M-1}$ states
- State probabilities $p(k_1, k_2, \ldots, k_M)$ satisfy balance equations ($c_m = 1$)

$$p(\underline{k}) \sum_{m=1}^{M} \mu_m \epsilon(k_m) = \sum_{n=1}^{M} \sum_{m=1}^{M} p(\underline{k} + \underline{e}_n - \underline{e}_m) \mu_n p_{nm} \epsilon(k_m)$$

where $\underline{e}_m = (0, \ldots, 1, \ldots, 0)$ with 1 at place m and $\epsilon(k) = \begin{cases} 1 & \text{if } k > 0 \\ 0 & \text{else} \end{cases}$

- State probabilities $p(k_1, k_2, \ldots, k_M)$ have product form

$$p(\underline{k}) = C p_1(k_1) p_2(k_2) \cdots p_M(k_M)$$

where C is normalizing constant and

$$p_m(k_m) = \left(\frac{v_m}{\mu_m}\right)^{k_m}, \quad k_m = 0, 1, \ldots$$

with v_m the “arrival rate” to workstation m: Again the product of $M/M/1$ solutions!

Exponential closed Single server network: Visiting frequency

- v_m is the relative arrival rate or visiting frequency to m

$$v_m = \sum_{n=1}^{M} v_n p_{nm}, \quad m = 1, \dots, M$$

- Equations determine v_m up to a multiplicative constant
- Set $v_1 = 1$: Then v_m is the expected number of visits to m in between two successive visits to station 1
- Product form result also valid for fixed routing
- Although $p(\underline{k})$ is again a product, the queues at stations are dependent!

Exponential closed Single server network: Normalizing constant

- Define

$$C(m,n) = \sum_{\substack{k_1,\ldots,k_m \geq 0 \\ \sum_{i=1}^m k_i = n}} \left(\frac{v_1}{\mu_1}\right)^{k_1} \left(\frac{v_2}{\mu_2}\right)^{k_2} \cdots \left(\frac{v_m}{\mu_m}\right)^{k_m}$$

- Interpretation: $C(m,n)$ is sum of products in network with stations $1,\ldots,m$ and population n
- Normalizing constant $C = \frac{1}{C(M,N)}$
- Question: How to calculate $C(M,N)$?
- Answer: Buzen's algorithm

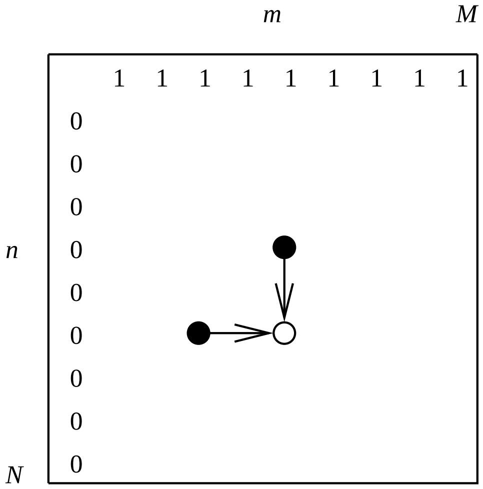
Exponential closed Single server network: Normalizing constant

- Buzen's algorithm:

$$C(m, n) = C(m-1, n) + \frac{v_m}{\mu_m} C(m, n-1)$$

with initial conditions

$$C(0, n) = 0, \quad n = 1, \dots, N, \quad C(m, 0) = 1, \quad m = 1, \dots, M$$



Exponential closed Single server network: Mean values

- Question: What is the real arrival rate λ_m?
- Answer:

$$\lambda_M = v_M \frac{C(M, N-1)}{C(M, N)}$$

and

$$\lambda_m = \frac{v_m}{v_M} \lambda_M$$

- Question: What is mean number $E(L_M)$ in station M?
- Answer:

$$E(L_M) = \frac{1}{C(M, N)} \sum_{k_M=0}^{N} k_M \left(\frac{v_M}{\mu_M}\right)^{k_M} C(M-1, N-k_M)$$

- Question: What is expected cycle time $E(C)$ between two visits to station 1?
- Answer: By Little's law

$$E(C) = \frac{N}{\lambda_1}$$

Exponential closed Multi server network

- State probabilities $p(k_1, k_2, \dots, k_M)$ have product form

$$p(\underline{k}) = Cp_1(k_1)p_2(k_2)\cdots p_M(k_M)$$

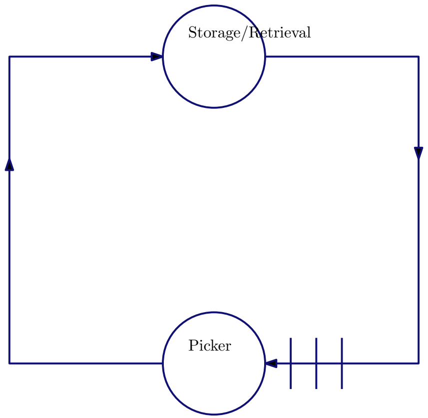
where C is normalizing constant and

$$p_m(k_m) = \prod_{k=1}^{k_m} \frac{v_m}{\mu_m(k)}$$

where $\mu_m(k) = \min(k, c_m)\mu_m$ and v_m visiting frequency to workstation m

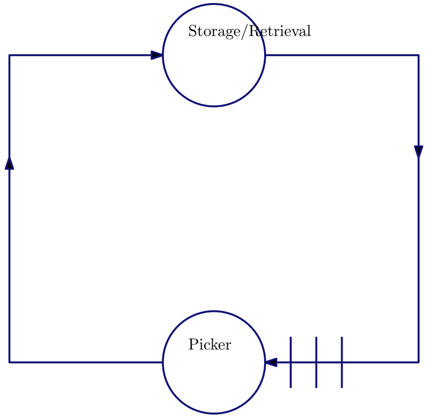
- Product of $M/M/c_m$ solutions with arrival rate v_m and service rate μ_m!
- Normalizing constant C can again be calculated via recursion

Example: KIVA system



- N circulating robots
- Pick station is Exponential single server with rate μ_P
- Storage/Retrieval station is Exponential infinite server with rate μ_{SR}
- Visiting frequency $v_1 = v_2 = 1$

Example: KIVA system



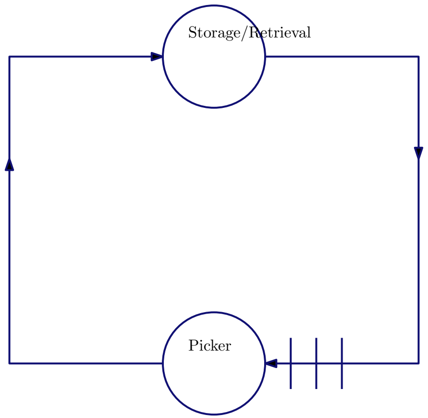
- State probabilities

$$p(k_P, k_{SR}) = p(N - k_{SR}, k_{SR}) = C\left(\frac{1}{\mu_P}\right)^{N-k_{SR}} \frac{1}{k_{SR}!}\left(\frac{1}{\mu_{SR}}\right)^{k_{SR}} = \frac{C}{\mu_P^N}\frac{1}{k_{SR}!}\left(\frac{\mu_P}{\mu_{SR}}\right)^{k_{SR}}, \quad k_{SR} = 0, 1, \ldots, N$$

- Normalizing constant

$$\frac{\mu_P^N}{C} = \sum_{k_{SR}=0}^{N} \frac{1}{k_{SR}!}\left(\frac{\mu_P}{\mu_{SR}}\right)^{k_{SR}}$$

Example: KIVA system



- Throughput

$$\lambda_{SR} = \lambda_P = \mu_P(1 - p(0, N))$$

Exponential closed Single server network: Arrival theorem

- Question: What is the state seen by job moving from one station to another?
- Answer: Number of jumps per time unit that see network in state $\underline{k} \in S(N-1) = \{\underline{k} \geq 0 | \sum_{i=1}^{M} k_i = N-1\}$

$$\sum_{m=1}^{M} p(\underline{k} + \underline{e}_m)\mu_m = \frac{1}{C(M,N)} p_1(k_1)\cdots p_M(k_M) \sum_{m=1}^{M} v_m$$

- Number of all jumps per time unit in network

$$\sum_{\underline{l} \in S(N-1)} \sum_{m=1}^{M} p(\underline{l} + \underline{e}_m)\mu_m = \frac{1}{C(M,N)} \sum_{\underline{l} \in S(N-1)} p_1(l_1)\cdots p_M(l_M) \sum_{m=1}^{M} v_m,$$

- Fraction of jumps per time unit that see network in state $\underline{k} \in S(N-1)$

$$\frac{\frac{1}{C(M,N)} p_1(k_1)\cdots p_M(k_M) \sum_{m=1}^{M} v_m}{\frac{1}{C(M,N)} \sum_{\underline{l} \in S(N-1)} p_1(l_1)\cdots p_M(l_M) \sum_{m=1}^{M} v_m} = \frac{1}{C(M,N-1)} p_1(k_1)\cdots p_M(k_M)$$

which is probability that network with $N-1$ circulating jobs is in state $\underline{k}$

Exponential closed Single server network: Arrival theorem

- Question: What is the state seen by job moving from one station to another?
- Answer: Job moving from one station to another sees network in equilibrium with population $N - 1$
- Remarks:
  - Also valid for Multi server networks
  - Also valid for jobs moving to specific station
- Question: What is the impact of the arrival theorem?
- Answer: Mean Value Analysis

Exponential closed Single server network: Mean value analysis

- Define for network with population k

$$\begin{aligned} \mathrm{E}(S_m(k)) &= \text{Mean sojourn time in station } m \\ \Lambda_m(k) &= \text{Throughput of station } m \\ \mathrm{E}(L_m(k)) &= \text{Mean number in station } m \end{aligned}$$

- For population $k = 1, 2, \dots, N$

$$\begin{aligned} \mathrm{E}(S_m(k)) &= \mathrm{E}(L_m(k-1))\frac{1}{\mu_m} + \frac{1}{\mu_m} \quad \text{(Arrival theorem)} \\ \Lambda_m(k) &= \frac{k v_m}{\sum_{n=1}^{M} v_n \mathrm{E}(S_n(k))} \quad \text{(Little)} \\ \mathrm{E}(L_m(k)) &= \Lambda_m(k)\mathrm{E}(S_m(k)) \quad \text{(Little)} \end{aligned}$$

with initially $E(L_m(0)) = 0$

Exponential closed Multi server network: Mean value analysis

- For population $k = 1, 2, \dots, N$

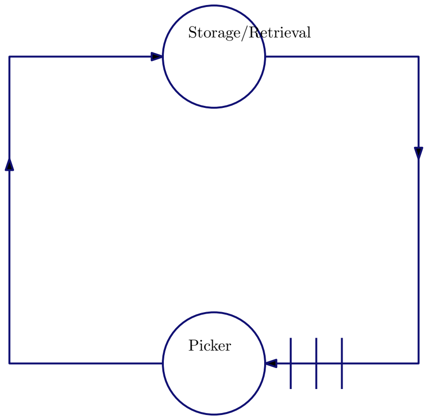
$$
\begin{aligned}
E(S_m(k)) &= \Pi_m(k-1)\frac{1}{c_m\mu_m} + \left(E(L_m(k-1)) - \frac{\Lambda_m(k-1)}{\mu_m}\right)\frac{1}{c_m\mu_m} + \frac{1}{\mu_m} \\
\Lambda_m(k) &= \frac{kv_m}{\sum_{n=1}^{M} v_n E(S_n(k))} \\
E(L_m(k)) &= \Lambda_m(k)E(S_m(k))
\end{aligned}
$$

where $\Pi_m(k-1)$ is probability that all servers are busy

- $\Pi_m(k-1)$ can be approximated by probability of waiting Π_W in $M/M/c_m$ with $\lambda = \Lambda_m(k-1)$ and $\mu = \mu_m$
- For $c_m = \infty$ (no waiting)

$$E(S_m(k)) = \frac{1}{\mu_m}$$

Example: KIVA system



- For population $k = 1, 2, \ldots, N$

$$
\begin{aligned}
E(S_P(k)) &= E(L_P(k-1))\frac{1}{\mu_P} + \frac{1}{\mu_P}, \quad E(S_{SR}(k)) = \frac{1}{\mu_{SR}} \\
\Lambda_P(k) &= \Lambda_{SR}(k) = \frac{k}{E(S_P(k)) + \frac{1}{\mu_{SR}}} \\
E(L_P(k)) &= \Lambda_P(k)E(S_P(k)) = k - E(L_{SR}(k))
\end{aligned}
$$

with initially $E(L_P(0)) = E(L_{SR}(0)) = 0$

General closed network: Approximate mean value analysis

- Service times is station m are General with mean $\mathrm{E}(B_m)$, cv c_{B_m} and mean residual $\mathrm{E}(R_m) = \frac{1}{2}(1+c_{B_m}^2)\mathrm{E}(B_m)$
- For population $k = 1, 2, \dots, N$

$$\begin{aligned}
\mathrm{E}(S_m(k)) &= \Pi_m(k-1)\frac{\mathrm{E}(R_m)}{c_m} + \big(\mathrm{E}(L_m(k-1)) - \Lambda_m(k-1)\mathrm{E}(B_m)\big)\frac{\mathrm{E}(B_m)}{c_m} + \mathrm{E}(B_m) \\
\Lambda_m(k) &= \frac{k v_m}{\sum_{n=1}^{M} v_n \mathrm{E}(S_n(k))} \\
\mathrm{E}(L_m(k)) &= \Lambda_m(k)\mathrm{E}(S_m(k))
\end{aligned}$$

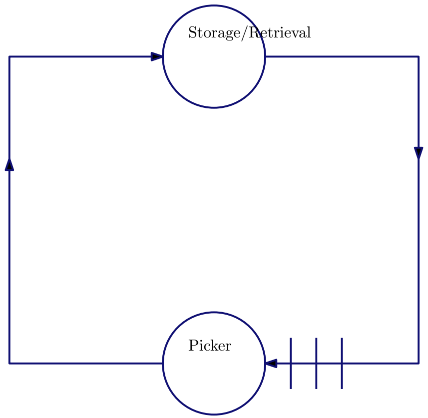
where $\Pi_m(k-1)$ can be approximated by probability of waiting in $M/M/c$

- In single server station

$$\mathrm{E}(S_m(k)) = \rho_m(k-1)\mathrm{E}(R_m) + (\mathrm{E}(L_m(k-1)) - \rho_m(k-1))\ \mathrm{E}(B_m) + \mathrm{E}(B_m)$$

where $\rho_m(k-1) = \Lambda_m(k-1)\ \mathrm{E}(B_m)$

Example: KIVA system



- General pick times and general storage/retrieval times
- For population $k = 1, 2, \ldots, N$

$$\mathrm{E}(S_P(k)) = \rho_P(k-1)\mathrm{E}(R_P) + (\mathrm{E}(L_P(k-1)) - \rho_P(k-1))\ \mathrm{E}(B_P) + \mathrm{E}(B_P), \quad \mathrm{E}(S_{SR}(k)) = \mathrm{E}(B_{SR})$$

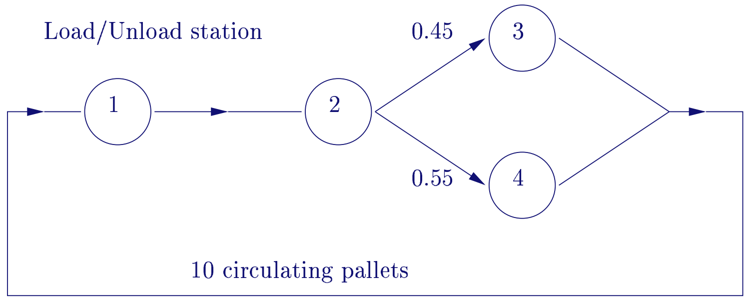
$$\Lambda_P(k) = \Lambda_{SR}(k) = \frac{k}{\mathrm{E}(S_P(k)) + \mathrm{E}(B_{SR})}$$

$$\mathrm{E}(L_P(k)) = \Lambda_P(k)\mathrm{E}(S_P(k)) = k - \mathrm{E}(L_{SR}(k))$$

with initially $\mathrm{E}(L_P(0)) = \mathrm{E}(L_{SR}(0)) = 0$ and $\rho_P(k) = \Lambda_P(k)\mathrm{E}(B_P)$

General closed network: Example

- Closed system with 4 single server stations and 10 circulating pallets



- Processing characteristics

Station	$E(B_m)$	$c^2_{B_m}$
1	1.25	0.25
2	1.25	0.50
3	2.00	0.33
4	1.60	1.00

General closed network: Example

- Mean value analysis: $\Lambda_1(10) = 0.736$ parts per time unit
- Simulation: $\Lambda_1(10) = 0.743 \pm 0.003$ parts per time unit
- Mean sojourn times

Station	$E(S_m(10))$	
	amva	sim
1	4.417	4.890 ± 0.106
2	5.050	4.760 ± 0.169
3	4.181	3.860 ± 0.068
4	4.086	3.790 ± 0.118

Exponential closed Multi-class network

- Workstations $1, \dots, M$
- Workstation m has c_m parallel identical machines
- R job types
- Population vector $\underline{N} = (N_1, N_2, \dots, N_R)$ where N_r is number of circulating jobs of type r
- Service times in workstation m are Exponential with rate μ_m (same for each job type)
- Service is in order of arrival
- Routing of type r jobs: type r job moves from workstation m to n with probability p^r_{mn}
- v_{mr} is relative visiting frequency to station m of type r jobs

$$v_{mr} = \sum_{n=1}^{M} v_{nr} p^r_{nm}, \qquad m = 1, 2, \dots, M$$

Exponential closed Multi-class network: Arrival theorem

- Type r job moving from one station to another sees network in equilibrium with population $\underline{N} - \underline{e}_r$

Exponential closed Multi-class network: Mean value analysis

- Define for network with population $\underline{k}$

$$\begin{aligned} E(S_{mr}(\underline{k})) &= \text{Mean sojourn time in work station } m \text{ for type } r \text{ job} \\ \Lambda_{mr}(\underline{k}) &= \text{Throughput of type } r \text{ jobs of station } m \\ E(L_{mr}(\underline{k})) &= \text{Mean number of type } r \text{ jobs in station } m \end{aligned}$$

- For population vectors $\underline{k} = \underline{0}$ to $\underline{k} = \underline{N}$

$$\begin{aligned} E(S_{mr}(\underline{k})) &= \sum_{s=1}^{r} E(L_{ms}(\underline{k} - \underline{e}_r)) \frac{1}{\mu_m} + \frac{1}{\mu_m} \\ \Lambda_{mr}(\underline{k}) &= \frac{N_r v_{mr}}{\sum_{n=1}^{M} v_{nr} E(S_{nr}(\underline{k}))} \\ E(L_{mr}(\underline{k})) &= \Lambda_{mr}(\underline{k}) E(S_{mr}(\underline{k})) \end{aligned}$$

with initially $E(L_{ms}(\underline{0})) = \underline{0}$

- Number of recursion steps is $\prod_{r=1}^{R} N_r$!

Exponential closed Multi-class network: Breaking the recursion

- Assume Type r job moving from one station to another sees network in equilibrium with population $\underline{N}$
- For population vector $\underline{N}$

$$\begin{aligned} \mathrm{E}(S_{mr}(\underline{N})) &= \sum_{s=1}^{r} \mathrm{E}(L_{ms}(\underline{N}))\frac{1}{\mu_m} + \frac{1}{\mu_m} \\ \Lambda_{mr}(\underline{N}) &= \frac{N_r v_{mr}}{\sum_{n=1}^{M} v_{nr}\mathrm{E}(S_{nr}(\underline{N}))} \\ \mathrm{E}(L_{mr}(\underline{N})) &= \Lambda_{mr}(\underline{N})\mathrm{E}(S_{mr}(\underline{N})) \end{aligned}$$

- Avoid self queueing

$$\mathrm{E}(S_{mr}(\underline{N})) = \sum_{s\neq r} E(L_{ms}(\underline{N}))\frac{1}{\mu_m} + \frac{N_r - 1}{N_r}\mathrm{E}(L_{mr}(\underline{N}))\frac{1}{\mu_m} + \frac{1}{\mu_m}$$

Exponential closed Multi-class network: Fixed point equations

- $3MR$ equations for $3MR$ unknowns $\mathrm{E}(S_{mr}(\underline{N}))$, $\Lambda_{mr}(\underline{N})$ and $\mathrm{E}(L_{mr}(\underline{N}))$

$$\mathrm{E}(S_{mr}(\underline{N})) = \sum_{s\neq r} \mathrm{E}(L_{ms}(\underline{N}))\frac{1}{\mu_m} + \frac{N_r - 1}{N_r}\mathrm{E}(L_{mr}(\underline{N}))\frac{1}{\mu_m} + \frac{1}{\mu_m}$$

$$\Lambda_{mr}(\underline{N}) = \frac{N_r v_{mr}}{\sum_{n=1}^{M} v_{nr}\mathrm{E}(S_{nr}(\underline{N}))}$$

$$\mathrm{E}(L_{mr}(\underline{N})) = \Lambda_{mr}(\underline{N})\mathrm{E}(S_{mr}(\underline{N}))$$

- Solution by successive substitutions