Generating functions (GF)

$p_n = \Pr\{X = n\}$, $n = 0, 1, \ldots$; $\sum_{n=0}^{\infty} p_n = 1$

$$P(z) := E[z^X] = \sum_{n=0}^{\infty} p_n z^n, \quad |z| \le 1$$

$$|P(z)| \le \sum p_n |z^n| \le \sum p_n = 1.$$

ex. • $p_n = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$, $n = 0, 1, \ldots \Rightarrow P(z) = e^{-\lambda(1-z)t}$

• $p_n = (1-\rho)\rho^n$, $n = 0, 1, \ldots \Rightarrow P(z) = \frac{1-\rho}{1-\rho z}$

GF uniquely determines prob. distr.

Laplace Stieltjes Transforms (LST)

$F(t) = Pr\{Y < t\}$, $t \geq 0$; $\lim_{t \to \infty} F(t) = 1$

$\varphi(s) := E\left[e^{-sY}\right] = \int_{t=0-}^{\infty} e^{-st}\, dF(t)$, $\mathrm{Re}\, s \geq 0$

$$|\varphi(s)| \leq \int_{t=0-}^{\infty} |e^{-st}|\, dF(t) \leq \int_{t=0-}^{\infty} dF(t) = 1$$

ex. $F(t) = 1 - e^{-\lambda t}$, $t \geq 0 \Rightarrow \varphi(s) = \dfrac{\lambda}{\lambda + s}$

LST uniquely determines prob. distr.

GF: moments

$$EX = \frac{d}{dz} P(z) \Big|_{z=1}$$

$$E[X(X-1)\dots(X-n+1)] = \frac{d^n}{dz^n} P(z) \Big|_{z=1}$$

ex Poisson: $EX = \lambda t$,
$EX^2 = \lambda t + \lambda^2 t^2$.

Sums of independent s.v.

$$E\left[z^{X_1+\dots+X_n}\right] = E\left[z^{X_1} \dots z^{X_n}\right]$$

$$= E[z^{X_1}] \dots E[z^{X_n}]$$

ex Poisson:

$$\left.\begin{array}{l} X \sim P(\lambda t) \\ V \sim P(\mu t) \\ X, V \text{ indep.} \end{array}\right\} \Rightarrow X + V \sim P((\lambda+\mu), t).$$

LST: moments

$$EY = -\frac{d}{ds}\varphi(s)\Big|_{s=0}$$

$$EY^n = (-1)^n \frac{d^n}{ds^n}\varphi(s)\Big|_{s=0}$$

ex. negative exponential distribution

$F(t) = 1 - e^{-\lambda t}, \quad t \geq 0.$

$EY = \frac{1}{\lambda}, \qquad EY^2 = \frac{2}{\lambda^2}$

Sums of independent s.v.

$$E[e^{-s(Y_1+\cdots+Y_n)}] = E[e^{-sY_1} \cdot \ldots \cdot e^{-sY_n}]$$

$$= E[e^{-sY_1}] \cdot \ldots \cdot E[e^{-sY_n}].$$

ex. neg. exp.: $Y_i \sim \exp(\lambda), \quad i = 1, \ldots, n \Rightarrow$

$\sum_{i=1}^{n} Y_i \sim \text{Erlang}(n, \lambda)$,

with LST $\left(\frac{\lambda}{\lambda+s}\right)^n$.

density: $\frac{\lambda^{n-1} t^{n-1}}{(n-1)!}\lambda e^{-\lambda t}, \quad t > 0.$

Random sums

$Y_1, Y_2, \ldots$ i.i.d. s.v., LST $\varphi(s)$

s.v. N independent of $Y_1, Y_2, \ldots$, GF $G(z)$

$$Z := \sum_{i=1}^{N} Y_i$$

$$E\left[e^{-sZ}\right] = \sum_{n=0}^{\infty} Pr\{N=n\}\, E\left[e^{-s\sum_1^N Y_i} \mid N=n\right]$$

$$= \sum_{n=0}^{\infty} Pr\{N=n\}\, \varphi^n(s)$$

$$= G(\varphi(s)), \qquad Re\ s \geq 0.$$

Rm.: $|\varphi(s)| \leq 1$ for $Re\ s \geq 0 \Rightarrow$
$G(\varphi(s))$ is defined for $Re\ s \geq 0$

ex.: $Y_i \sim \exp(\lambda)$, $N \sim geom(p) \Rightarrow$

$Z \sim \exp(\lambda(1-p))$

ex.: $EZ = EY_1 \, EN$,

$$Var(Z) = Var(N)(EY_1)^2 + EN\ Var(Y_1)$$

Feller's convergence theorem for LST

$F_n(t)$, $n=1,2,\ldots$: prob. distr., $F_n(t)=0$ for $t<0$.

If $\forall s>0$,

$$\varphi_n(s) = \int_{0-}^{\infty} e^{-st}\, dF_n(t) \xrightarrow[n\to\infty]{} \varphi(s),$$

then $\varphi(s)$ is the LST of a prob. distr. $F(\cdot)$,

and $F_n(t) \to F(t)$ in each continuity point of $F(t)$.

$F(\cdot)$ is non-defective $\Leftrightarrow \lim_{s\downarrow 0} \varphi(s)=1$.

ex. Erlang $(n, n\lambda) \longrightarrow$ Det $(\frac{1}{\lambda})$.

ex. (discrete version):

$$\text{bin}(n, \tfrac{\lambda}{n}) \longrightarrow P(\lambda)$$