Exercises Introduction to Stochastic Processes, week 1

Exercises 10-13 are particularly representative of exam exercises.

Exercise 1
Let X be a non-negative random variable with $\mathbf{E}[X^2] < \infty$, having probability density function $f(\cdot)$. The following formula was derived during the lectures:

$$\mathbf{E}[X] = \int_0^\infty \mathbf{P}(X > u)\mathrm{d}u.$$

Show that

$$\mathbf{E}[X^2] = \int_0^\infty 2u\mathbf{P}(X > u)\mathrm{d}u.$$

Exercise 2
Let X_i be exponentially distributed with parameter $\lambda_i, i = 1, 2$.

(a) Compute $\mathbf{P}(X_1 > u + v \mid X_1 > u)$.

(b) What is the probability that $X_1 \leq X_2$?

(c) What is the distribution of $\min\{X_1, X_2\}$?

Exercise 3
Consider two parallel processors, 1 and 2. Job A_i is being processed by processor $i = 1, 2$. Processing times of jobs by processor i are exponentially distributed with intensity λ_i. Job A_3 is waiting in line and will be processed by the processor that first completes its current job.

(a) Let $\lambda_1 = \lambda_2 = \lambda$. Without calculations, derive the probability that job A_3 is the last of the three jobs to be completed.

(b) Compute the probability that job A_3 is the last of the three jobs to be completed for arbitrary λ_1 and λ_2.

Exercise 4 (The hyper-exponential distribution)
Let X_i be exponentially distributed with parameter $\lambda_i, i = 1, 2$. Suppose B is

a random variable with $\mathbf{P}(B = 1) = p$ and $\mathbf{P}(B = 2) = 1 - p$, with $0 < p < 1$. Let Y be defined as follows: $Y = X_1$ if $B = 1$ and $Y = X_2$ if $B = 2$. Calculate $\mathbf{P}(Y > x)$ by conditioning on the values of B. Consequently, compute the density of Y.

Exercise 5

Let $X_1, X_2, \ldots$ be a sequence of independent, exponentially distributed random variables with common parameter λ.

(a) Compute the density of $X_1 + X_2$.

(b) Using induction on n, show that the density of $S_n = \sum_{i=1}^{n} X_i$ is given by

$$\lambda \mathrm{e}^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}.$$

Remark: This is the density of the *Erlang* distribution with parameters n and λ.

(c) Verify that the above density is the derivative with respect to t of the Erlang distribution given in Ross.

Hint: Use the following property: If X and Y are independent, non-negative random variables with densities f and g, respectively, then the density of $X+Y$ equals $\int_0^t f(u)g(t-u)\mathrm{d}u$.

Exercise 6

Consider a Markov chain with transition probabilities $p_{12} = 1, p_{21} = 1/2, p_{23} = 1/2, p_{32} = 1/2$ and $p_{33} = 1/2$.

(a) Determine the steady-state distribution of this Markov chain.

(b) Determine the expected time it takes the process to return to state i (starting from i), for $i = 1, 2, 3$.

Exercise 7

Consider a Markov chain with transition matrix

$$P = \begin{bmatrix} 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $X_n \in \{1, 2, 3, 4\}$ be the state of the chain at time $n = 0, 1, 2, \ldots$. Determine $\lim_{n\to\infty} P(X_n = k | X_0 = 1\}$, $k = 1, 2, 3, 4$.

Exercise 8
The home page of the extremely popular T.E. Acher is regularly consulted by many students worldwide. New 'visits' to the home page by students occur according to a Poisson process with an average of 10 students per hour. Mr. Acher is also highly respected by colleagues. The average number of colleagues that visit the home page per hour is 2 (also according to a Poisson process).
(a) What is the probability that the home page is visited at least twice during one hour?
(b) What is the probability that the home page is not visited at all over the course of 15 minutes?

A student that visits the home page 'clicks' on the link to an overview of Mr. Acher's research activities with probability 2/5.
(c) Determine the probability that during one working day (8 hours) exactly 1 student consults the research overview.

Exercise 9
A processor is inspected weekly in order to determine its condition. The condition of the processor can either be *perfect, good, reasonable* or *bad*. A new processor is still perfect after one week with probability 0.7, with probability 0.2 the state is good and with probability 0.1 it is reasonable. A processor in good conditions is still good after one week with probability 0.6, reasonable with probability 0.2 and bad with probability 0.2. A processor in reasonable condition is still reasonable after one week with probability 0.5 and bad with probability 0.5. A bad processor must be repaired. The reparation takes one week, after which the processor is again in perfect condition.
(a) Formulate a Markov chain that describes the state of the machine and draw the transition probabilities in a network.
(b) Determine the steady-state probabilities of the Markov chain.

Exercise 10
Consider the Markov Chain $(X_n)_{n\geq 0}$ with state space $\{1,2,3,4,5,6\}$ and transition matrix:

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{5} & \frac{1}{5} & 0 & 0 \end{pmatrix} \tag{1}$$

a) Draw the network of possible transitions and determine the classes of communicating states.
b) Verify – for all possible initial states – whether there exists a limiting distribution and, if so, determine this distribution.

c) Determine the probability that state 1 is ever reached, starting in state 2.

Exercise 11
Two types of consultations occur at a database according to two independent Poisson processes: “read” consultations arrive at rate λ_R and ”write” consultations at rate λ_W.
a. What is the probability that the time interval between two consecutive ”read” consultations is larger than t?
b. What is the probability that the first next arriving consultation is a ”read” consultation?
c. What is the probability that during the time interval $[0, t]$ at most three ”write” consultations arrive?
d. What is the probability that during the time interval $[0, t]$ at least two consultations arrive?
e. Determine the distribution of the number of arrived ”read” consultations during $[0, t]$, given that in this interval a total number of n consultations occurred.

Exercise 12
Consider the route of a drunk visiting a collection of $N + 1$ bars numbered $0, 1, \ldots, N - 1, N$. From bar i the drunk walks to bar $i + 1$ with probability p and to bar $i - 1$ with probability $q = 1 - p$, $i = 1, \ldots, N - 1$. After visiting bar 0 the drunk always goes to bar 1 and after bar N the drunk visits bar $N - 1$. X_n denotes the position of the drunk after the nth walk.
a) Determine the expected number of walks to reach bar 3 from bar 0.
b) Assume $p = q = 0.5$. Determine the steady-state distribution of the Markov chain $(X_n)_{n\geq 0}$.

Exercise 13
The number of orders being processed at a factory can be described by a Markov chain with state space $\{0, 1, 2, 3, \ldots\}$. For a given positive integer N, the transition probabilities are $P_{0,0} = 0.5$, $P_{i,i+1} = 0.5$ for $i = 0, \ldots, N$, $P_{i,i+1} = p$ for $i = N + 1, N + 2, \ldots$, $P_{i,i-1} = 0.5$ for $i = 1, \ldots, N$ and $P_{i,i-1} = 1 - p$ for $i = N + 1, N + 2, \ldots$. Here, $p < 0.5$.
(a) Why is this Markov chain irreducible, aperiodic and positive recurrent?
(b) Determine the limiting distribution of the Markov chain.
(c) Let $N = 3$. What is the expected number of transitions to reach state 3 from state 0?