LNMB EXAM Introduction to Stochastic Processes (ISP)
Saturday October 7, 2006, 11.00-14.00 hours.

The exam consists of five exercises, each with several parts. For each part it is indicated between square brackets how many points can be earned for the item; the total number of points is 40.

EXERCISE 1

The Markov chain $(X_n)_{n\geq 0}$ with state space $\{1,2,3,4,5,6\}$ has as matrix of transition probabilities:

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\ \frac{2}{5} & \frac{1}{5} & 0 & 0 & \frac{2}{5} & 0 \\ \frac{2}{5} & \frac{1}{2} & 0 & \frac{1}{10} & 0 & 0 \\ 0 & 0 & \frac{1}{10} & 0 & \frac{1}{2} & \frac{2}{5} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

a) [3 pt.] What are the classes of communicating states?
b) [3 pt.] What is the probability of ever reaching state 1 if the process starts in state 4?
c) [4 pt.] Indicate for each initial state whether a limiting distribution exists, and if yes, determine it.

EXERCISE 2

A processor can work on two types of tasks. Type-1 tasks arrive according to a Poisson process with a rate of 100 per second and type-2 tasks according to a Poisson process with a rate of 200 per second. The two arrival processes are independent. Both types of tasks have exponentially distributed service times, with a mean of 5 milliseconds for type 1 and 2 milliseconds for type 2.
a. [2 pt.] What is the probability that during 10 milliseconds no new tasks arive?
b. [2 pt.] What is the probability that the first next task to arrive is of class 1?
c. [2 pt.] What is the probability that during 20 milliseconds at least two tasks of type 1 arrive?
d. [2 pt.] Determine the Laplace-Stieltjes transform of the time until K tasks have arrived, and use this expression to obtain the variance of that time.
e. [2 pt.] Determine the generating function of the number of arrivals during the service time of an arbitrary type-1 task.

EXERCISE 3
Consider a branching process $\{X_i, i = 0, 1, 2, \ldots\}$, starting with one particle: $X_0 = 1$. The number of offspring Z of one particle is distributed as

$$P(Z = i) = (1 - p)^i p, \qquad i = 0, 1, 2, \ldots$$

a. [2 pt.] Give $E[Z]$ and determine $E[X_i]$ for $i = 1, 2, \ldots$.
b. [2 pt.] Determine the probability that the population dies out (for every $0 \leq p \leq 1$).

EXERCISE 4
Two workers share an office that contains two telephones. At any time, each worker is either 'working' or 'on the phone'. Each 'working' period of a worker lasts for an exponentially distributed time with rate λ, and each 'on the phone' period lasts for an exponentially distributed time with rate μ. Let $X(t)$ denote the number of workers 'on the phone' at time t.
a. [2 pt.] Argue that $\{X(t), t \geq 0\}$ is a continuous time Markov chain and give its transition rates.
b. [2 pt.] Suppose that at time 0 there is exactly one worker 'on the phone'. Compute the expected time until both workers are 'working' for the first time.
c. [2 pt.] What proportion of time are both workers 'working'?
Suppose now that one of the phones has broken down. Suppose that a worker who is about to use a phone and finds it being used begins a new 'working' period.
d. [2 pt.] What proportion of time are both workers 'working'?

EXERCISE 5
The lifetime of a machine is exponential with a mean of 1 year. When the machine fails, an emergency unit automatically takes over the production. The unit breaks down after exactly 2 months. Then the whole system (i.e., the machine and the emergency unit) is immediately replaced by a new one. This so-called unplanned replacement costs 1400 euro.
a. [2 pt.] At what rate is the system replaced?
b. [2 pt.] Determine the long-run average cost per year.
Now, to prevent unplanned replacements, the system is regularly inspected by a repairman. An inspection costs 200 euro. If the machine is still working, nothing is done (the system is still as good as new); otherwise the repairman immediately replaces the whole system, but this preventive replacement is cheaper than an unplanned replacement, namely 600 euro. Assume that the repairman always comes along exactly 10 months after the last event; an event is either an inspection or an unplanned replacement.
c. [2 pt.] What is the mean time between two successive events?
d. [2 pt.] Compute the long-run average cost per year.