LNMB EXAM Introduction to Stochastic Processes (ISP)
Tuesday September 23, 2008, 14.00-17.00 hours.

The exam consists of five exercises, each with several parts. For each part it is indicated between square brackets how many points can be earned for the item; the total number of points is 40.

EXERCISE 1

The Markov chain $(X_n)_{n\geq 0}$ with state space $\{1, 2, 3, 4, 5, 6\}$ has as matrix of transition probabilities:

$$\begin{pmatrix} \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \end{pmatrix}$$

a. [2 pt.] What are the classes of communicating states?
b. [3 pt.] Indicate for each initial state whether a limiting distribution exists, and if yes, determine it.
c. [3 pt.] Calculate, in the case of initial state 5, the probability that state 2 is ever reached.
d. [2 pt.] Determine the mean number of steps to reach state 2, starting in state 6.

EXERCISE 2

During summer thunderstorms occur according to a Poisson process with intensity $\lambda = 2$ per week.
a. [1 pt.] What is the expected number of storms in four weeks?
b. [2 pt.] What is the probability of having more than one storm during the first week of July?
c. [3 pt.] What is the probability that there is at least one storm every week during a period of 8 consecutive weeks?
d. [4 pt.] Each thunderstorm is either severe or non-severe. Severe thunderstorms occur according to a Poisson process with intensity $\mu = 0.5$ per week. What is the probability that the first severe thunderstorm occurs before the first non-severe thunderstorm?

EXERCISE 3

Consider a branching process $\{X_i, i = 0, 1, 2, \ldots\}$, starting with one particle: $X_0 = 1$. The probability generating function of the number of offspring Z of one particle is given by

$$G(r) = E[r^Z] = \frac{1-c}{1-cr}, \qquad |r| \leq 1,$$

where $0 < c < 1$.

a. [2 pt.] Give $E[Z]$ and determine $E[X_i]$ for $i = 1, 2, \ldots$.

b. [2 pt.] Compute the extinction probability for all possible values of c.

EXERCISE 4

In a helpdesk, two operators, denoted by A and B, are servicing incoming calls. The service times are exponential. The mean service time of A is 30 seconds and the mean service time of B is 1 minute. Calls arrive according to a Poisson stream with rate 2 calls per minute. An incoming call is routed to an idle operator; if both operators are idle, the call is assigned to either operator with equal probability. If both operators are busy, the incoming call is lost. Let $X(t)$ denote the state of the operators at time t; the possible states are 0 (both are idle), A (only A is busy), B and AB (both are busy).

a. [1 pt.] Argue that $\{X(t), t \geq 0\}$ is a continuous time Markov chain and give its transition rates.

b. [2 pt.] Suppose that at time 0 operator A is busy and B is idle. Compute the expected time until both operators are idle for the first time.

c. [2 pt.] Determine the equilibrium probabilities p_0, p_A, p_B and p_{AB}.

d. [1 pt.] What is the fraction of calls that is lost?

Now suppose that, when both operators are busy, it is possible to keep one incoming call 'on hold' until one of them becomes available. While this call is 'on hold', newly arriving calls are lost.

e. [2 pt.] What is now the fraction of calls that is lost?

EXERCISE 5

At the beginning of every week (40 hours) exactly 5 orders are released in a production unit. Orders are processed one by one; the processing time of an order is exponential with mean 8 hours. If all 5 orders have been processed before the end of the week, the unit will idle (until the start of the next week when a new batch of 5 orders is released). Idling costs 50 euros per hour. However, orders that have not been (completely) processed by the end of the period, are immediately outsourced (i.e., processed somewhere else). Outsourcing costs 100 euros per order.

a. [2 pt.] Compute the probability that i orders are processed during a week, $i = 0, 1, \ldots, 5$.

b. [2 pt.] Determine the mean number of orders processed per hour.

c. [2 pt.] What is the mean time the facility is idling during a week?

d. [2 pt.] Compute the long-run average cost per hour.