LNMB EXAM Introduction to Stochastic Processes (ISP)
Friday 19 November, 2010, 13.00-16.00 hours.

The exam consists of six exercises, each with several parts. For each part it is indicated between square brackets how many points can be earned for the item; the total number of points is 40.

EXERCISE 1

The Markov chain $(X_n)_{n\geq 0}$ with state space $\{1,2,3,4,5,6\}$ has as matrix of transition probabilities:

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{9} & 0 & \frac{1}{3} & \frac{5}{9} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a. [2 pt.] What are the classes of communicating states?
b. [2 pt.] Determine the period of each state.
c. [2 pt.] Indicate for each initial state whether a limiting distribution exists, and if yes, determine it.
d. [2 pt.] Calculate, in case of initial state 2, the probability that state 3 is ever reached.
e. [2 pt.] If the Markov chain starts in state 4, then what is the proportion of time the Markov chain spends in state 1?

EXERCISE 2

Cars arrive at a shopping center according to a Poisson process with rate λ. There are two parking lots at the shopping center, parking lot P_1 and P_2. An arriving car is parked at P_1 with probability p_1 and at P_2 otherwise.
a. [2 pt.] What is the probability that at least 5 cars have arrived at parking lot P_1 in $[0,t]$?
b. [2 pt.] What is the probability that at least 5 cars have arrived at P_1 in $[0,t]$, given that in $[0,t]$ no car arrived at parking lot P_2?
c. [2 pt.] What is the probability that exactly 5 cars have arrived at parking lot P_1 before the first car arrived at P_2?
d. [3 pt.] Suppose that exactly 5 cars have arrived at P_1 in $[0,10]$. Each of these cars stays (exactly) 7 time units at P_1. What is the probability that 3 of these cars are still present at time 10?

EXERCISE 3

Consider a branching process $\{X_i, i = 0, 1, 2, \ldots\}$, starting with one particle: $X_0 = 1$. The probability generating function of the number of offspring Z of one particle is given by

$$G(r) = E[r^Z] = \left(\frac{1-p}{1-pr}\right)^2, \qquad |r| \leq 1,$$

where $0 < p < 1$.

a. [2 pt.] Give $E[Z]$ and $\text{var}[Z]$ and determine $E[X_i]$ for $i = 1, 2, \ldots$.

b. [2 pt.] Determine the probability that the population dies out for $p = \frac{1}{4}$ and $p = \frac{1}{2}$.

EXERCISE 4

A job shop consists of three machines. The amount of time a machine works before breaking down is exponentially distributed with rate 2. The machines are fixed by two repairmen, in order of breakdown. If a machine breaks down and both of them are idle, then either repairman will be assigned to the machine with probability $\frac{1}{2}$. The amount of time it takes a single repairman to fix a machine is exponentially distributed with rate 3. Let $X(t)$ denote the number of machines in use at time t.

a. [1 pt.] Argue that $\{X(t), t \geq 0\}$ is a continuous time Markov chain and give its transition rate diagram.

b. [2 pt.] Suppose that at time 0 all three machines are working. Compute the expected time until both repairmen are busy for the first time.

c. [2 pt.] What is the average number of machines in use?

d. [2 pt.] What proportion of time is a repairman busy?

EXERCISE 5

Events occur according to a Poisson process with rate λ. Any event that occurs within a time 1 of the event that immediately preceded it is called a *red*-event.

a. [2 pt.] What proportion of all events are *red*-events?

b. [2 pt.] At what rate do *red*-events occur?

EXERCISE 6

The lifetime of a part is uniform on $(0, 1)$. If the part breaks down, it is immediately replaced by a new one, for which a cost of 4 is incurred.

a. [1 pt.] Compute the long-run average cost.

Now the part is preventively replaced by a new one as soon as it reaches the age of T (< 1). The preventive replacement cost is 1.

b. [2 pt.] Compute the long-run average cost.

c. [3 pt.] What value of T minimizes the long-run average cost?