LNMB EXAM Introduction to Stochastic Processes (ISP)
Monday September 27, 2010, 13.15-16.15 hours.

The exam consists of five exercises, each with several parts. For each part it is indicated between square brackets how many points can be earned for the item; the total number of points is 40.

EXERCISE 1

The Markov chain $(X_n)_{n\geq 0}$ with state space $\{1,2,3,4,5,6\}$ has as matrix of transition probabilities:

$$\begin{pmatrix} 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix} \quad (1)$$

a. [2 pt.] What are the classes of communicating states?
b. [3 pt.] Indicate for each initial state whether a limiting distribution exists, and if yes, determine it.
c. [2 pt.] Calculate, in the case of initial state 6, the probability that state 2 is ever reached.
d. [2 pt.] If the Markov chain starts in 2, then what is the mean time between visits to that state?

EXERCISE 2

Customers and robbers individually arrive at a bank according to two independent Poisson processes, with rates λ_c and λ_r.
a. [2 pt.] What is the probability that at least 3 persons have arrived in $[0, t]$?
b. [2 pt.] What is the probability distribution of the time until the third arrival?
c. [2 pt.] What is the probability that exactly 100 customers have arrived before the first bank robber?
d. [3 pt.] Suppose that exactly one bank robber has arrived in $[0, 1]$. The robber stays an exponentially distributed amount of time, with mean one. What is the probability that she is still present at time 1?

EXERCISE 3

Consider a branching process $\{X_i, i = 0, 1, 2, \ldots\}$, starting with one particle: $X_0 = 1$. The number of offspring Z of one particle is distributed as

$$P(Z = i) = (i + 1)(1 - p)^i p^2, \quad i = 0, 1, 2, \ldots,$$

where $0 < p < 1$.

a. [2 pt.] Give $E[Z]$ and determine $E[X_i]$ for $i = 1, 2, \ldots$.

b. [2 pt.] Determine the probability that the population dies out (for every $0 < p < 1$).

EXERCISE 4

In a helpdesk, two operators are servicing incoming calls. The service times are exponential with mean 30 seconds. Calls arrive according to a Poisson stream with rate 2 calls per minute. Incoming calls are handled in order of arrival. Let $X(t)$ denote the number of calls in the system (waiting and in service) at time t.

a. [1 pt.] Argue that $\{X(t), t \geq 0\}$ is a continuous time Markov chain and give its transition rate diagram.

b. [2 pt.] Suppose that at time 0 exactly one operator is busy. Compute the expected time until both operators are busy for the first time.

c. [2 pt.] Determine the limiting probabilities p_n of n calls in the system, $n = 0, 1, 2, \ldots$.

A period during which, uninterruptedly, both operators are occupied is called a crowded period. A period during which, uninterruptedly, at least one operator is idle is called a quiet period.

d. [2 pt.] How long lasts a quiet period on average?

e. [2 pt.] And how long lasts a crowded period on average?

EXERCISE 5

Consider a population for which new members are born according to a Poisson process with constant rate λ. So the birth rate does not depend on the population size. After a random amount of time, which is independent of the population size, a disaster occurs and the whole population (if there is any) immediately dies out. Then the population starts to grow all over again, that is, after an exponential time with rate λ the first member is born, and so on until the next disaster occurs. Disasters occur according to a Poisson process with rate μ.

a. [3 pt.] Compute the long-run fraction of time the population consists of n members, $n \geq 0$.

b. [2 pt.] Determine the mean population size just before a disaster.

c. [2 pt.] What is the long-run average population size?

d. [2 pt.] Determine the distribution of the population size just before a disaster.