LNMB EXAM Introduction to Stochastic Processes (ISP)
Saturday October 7, 2006, 11.00-14.00 hours.

EXERCISE 1
a. [3 pt.] $\{1,2,5\}$, $\{3,4\}$, $\{6\}$.
b. [3 pt.] Let $T_{ij} = P(\textit{ever in } j|X_0 = i)$. Then $T_{41} = \frac{1}{10}T_{31} + \frac{1}{2}1 + \frac{2}{5}0$ and $T_{31} = \frac{2}{5} + \frac{1}{2} + \frac{1}{10}T_{41}$. Hence $T_{31} = \frac{95}{99}$ and $T_{41} = \frac{59}{99}$.
c. [4 pt.] Starting from $\{1,2,5\}$: $V_1 = \{3/10, 5/10, 0, 0, 2/10, 0\}$.
Starting from 6: $V_6 = \{0,0,0,0,0,1\}$.
Starting from 3: $\frac{95}{99}V_1 + \frac{4}{99}V_6$.
Starting from 4: $\frac{59}{99}V_1 + \frac{40}{99}V_6$.

EXERCISE 2
a. [2 pt.] $\mathrm{e}^{-(200+100)*0.01} = \mathrm{e}^{-3}$.
b. [2 pt.] $\frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{1}{3}$.
c. [2 pt.] $1 - \mathrm{e}^{-\lambda_1 t} - \lambda_1 t\mathrm{e}^{-\lambda_1 t} = 1 - 3\mathrm{e}^{-2}$.
d. [2 pt.] $\frac{\lambda}{\lambda+s}$ is the LST of the time until one task has arrived, so $(\frac{\lambda}{\lambda+s})^K$ is the LST of the time until K tasks have arrived. Now use that $E[T_K^i]$ equals $(-1)^i$ times the i-th derivative of the latter LST to derive that the variance equals $\frac{K}{90000}$.
e. [2 pt.] $\int_0^\infty \mu_1 \mathrm{e}^{-\mu_1 t} \sum_{n=0}^\infty \mathrm{e}^{-\lambda_2 t}\frac{(\lambda_2 t)^n}{n!} z^n \mathrm{d}t = \frac{\mu_1}{\mu_1+\lambda_2(1-z)}$ with $\mu_1 = \lambda_2 = 200$.

EXERCISE 3
a. [2 pt.] $E[Z] = \frac{1-p}{p}$ and $E[X_i] = E[Z]^i = \frac{(1-p)^i}{p^i}$.
b. [2 pt.] Clearly $\pi_0 = 1$ for $\frac{1}{2} \leq p \leq 1$ (since then $E[Z] \leq 1$). If $0 \leq p < \frac{1}{2}$, then π_0 is the root in $[0,1)$ of

$$\pi_0 = \sum_{i=0}^{\infty} P[Z = i]\pi_0^i = \frac{p}{1-(1-p)\pi_0},$$

yielding

$$\pi_0 = \frac{p}{1-p}.$$

EXERCISE 4
a. [2 pt.] The transition rate matrix Q is

$$\begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -\lambda-\mu & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix}.$$

b. [2 pt.] Let T_i $(i = 1, 2)$ denote the time, starting from state i, it takes to enter state 0.

Then

$$E[T_1] = \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}E[T_2], \qquad E[T_2] = \frac{1}{2\mu} + E[T_1].$$

Solving these equations gives

$$E[T_1] = \frac{2\mu+\lambda}{2\mu^2}.$$

c. [2 pt.] The balance equations are:

$$\begin{aligned} 2\lambda P_0 &= \mu P_1, \\ \lambda P_1 &= 2\mu P_2. \end{aligned}$$

from which, together with the normalization equation, follows that

$$P_0 = \left(\frac{\mu}{\lambda+\mu}\right)^2.$$

Alternatively, this can be obtained by noting that both workers are working independently, and the probability that a working is 'working' is $\mu/(\lambda+\mu)$.
d. [2 pt.] Now only two states are possible, state 0 and 1. Balance of flow gives

$$2\lambda P_0 = \mu P_1,$$

and thus

$$P_0 = \frac{\mu}{\mu+2\lambda}.$$

EXERCISE 5
a. [2 pt.] The mean time between replacements is $\frac{7}{6}$ year. Hence the system is replaced $\frac{6}{7}$ times per year.
b. [2 pt.] The long-run average cost per year is $1400 \cdot \frac{6}{7} = 1200$ euro.
c. [2 pt.] The mean time between two events is

$$\int_{t=0}^{\frac{2}{3}} (t+\frac{1}{6})e^{-t}dt + \frac{5}{6}\int_{t=\frac{2}{3}}^{\infty} e^{-t}dt = \frac{7}{6} - e^{-2/3} = 0.653.$$

d. [2 pt.] The mean cost of an event is $1400 \cdot P[L < \frac{2}{3}] + 200 \cdot P[L > \frac{2}{3}] + 600 \cdot P[\frac{2}{3} < L < \frac{5}{6}] = 831$ euro. Hence the mean cost per year is

$$\frac{831}{0.653} = 1273 \text{ euro}.$$