LNMB EXAM Introduction to Stochastic Processes (ISP)
Tuesday September 23, 2008, 14.00-17.00 hours.

EXERCISE 1
a. [2 pt.] Classes of communicating states are $A = \{1, 4\}$, $B = \{5\}$ and $C = \{2, 3, 6\}$.
b. [3 pt.] For initial state in A the limiting distribution fulfills

$$\begin{aligned}\pi_1 &= \tfrac{2}{3}\pi_1 + \pi_4 \\ \pi_4 &= \tfrac{1}{3}\pi_1.\end{aligned}$$

It follows from $\pi_1 + \pi_4 = 1$ that $\pi_1 = 3/4$ and $\pi_4 = 1/4$ ($\pi_i = 0$ for $i \notin A$). For initial state in the class C we have to solve

$$\begin{aligned}\pi_2 &= \pi_3 + \tfrac{1}{4}\pi_6 \\ \pi_3 &= \tfrac{3}{4}\pi_6 \\ \pi_6 &= \pi_2\end{aligned}$$

Together with $\pi_2 + \pi_3 + \pi_6 = 1$ this leads to $\pi_6 = \pi_2 = 4/11$ and $\pi_3 = 3/11$ ($\pi_i = 0$ for $i \notin C$). Finally, state 5 is transient and with probability $(1/6)/(5/6) = 1/5$ the class A is ever reached, with probability 4/5 the class C is ever reached. It follows that the limit distribution is given by

$$\begin{aligned}\pi_1 &= \tfrac{3}{4} \cdot \tfrac{1}{5} = \tfrac{3}{20}. \\ \pi_2 &= \tfrac{4}{11} \cdot \tfrac{4}{5} = \tfrac{16}{55}. \\ \pi_3 &= \tfrac{3}{11} \cdot \tfrac{4}{5} = \tfrac{12}{55}. \\ \pi_4 &= \tfrac{1}{4} \cdot \tfrac{1}{5} = \tfrac{1}{20}. \\ \pi_5 &= 0 \\ \pi_6 &= \tfrac{4}{11} \cdot \tfrac{4}{5} = \tfrac{16}{55}.\end{aligned}$$

c. [3 pt.] The probability that state 2 is ever reached from 5 is equal to the probability that the first step out of class C leads into class C, which is $\frac{4}{5}$ (see b.).
d. [2 pt.] The process can go from 6 to 2 either directly (w.p. $p = 1/4$) or via state 3 (w.p. $q = 3/4$). Hence the mean number of required steps is

$$\frac{1}{4} \cdot 1 + \frac{3}{4} \cdot 2 = \frac{7}{4}.$$

EXERCISE 2
Let $N(t)$ denote the number of thunderstorms during t weeks.
a. [1 pt.] $E[N(t)] = \lambda \cdot t = 2 \cdot 4 = 8$ thunderstorms in four weeks.
b. [2 pt.] $P[N(t) = k] = \frac{(2t)^k}{k!} e^{-2t}$, so that

$$P[N(1) > 1] = 1 - P[N(1) = 0] - P[N(1) = 1] = 1 - e^{-2} - 2e^{-2} \approx 0,594.$$

c. [3 pt.] Using the independent increment property, we have

$$\begin{aligned} P[\text{at least one storm every week, 8 weeks}] &= P[N(1) \geq 1]^8 = (1 - P[N(1) = 0])^8 \\ &= (1 - e^{-2})^8 \approx 0,313. \end{aligned}$$

d. [4 pt.] Non-severe thunderstorms occur with intensity $\nu = 3/2$ per week. It follows that the first thunderstorm is severe with probability $p = \frac{\mu}{\mu+\nu} = 1/4$. Another way of showing this is the following: the distributions of the time of the first severe (the first non-severe) thunderstorm is exponential with mean $1/\mu$ $(1/\nu)$. Then p is the probability that an exponential random variable with mean $1/\mu$ is smaller than the other, which is given by $\mu/(\mu + \nu)$.

EXERCISE 3

a. [2 pt.] $E[Z] = G'(1) = \frac{c}{1-c}$ and $E[X_i] = E[Z]^i = \left(\frac{c}{1-c}\right)^i$.

b. [2 pt.] The extinction probability $\pi_0 = 1$ for all c for which $E[Z] \leq 1$, and thus for all $0 < c \leq \frac{1}{2}$. If $\frac{1}{2} < c < 1$, then π_0 is the unique root on $(0, 1)$ of the equation

$$\pi_0 = G(\pi_0),$$

yielding

$$\pi_0 = \frac{1-c}{c}.$$

EXERCISE 4

Take as 1 minute as time unit.

a. [1 pt.] The state space is $\{0, A, B, AB\}$ with transition rate matrix

$$Q = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 2 & -4 & 0 & 2 \\ 1 & 0 & -3 & 2 \\ 0 & 1 & 2 & -3 \end{pmatrix}.$$

b. [2 pt.] Let T_A, T_B, T_{AB} denote the time, starting from state A, B and AB respectively, it takes to enter state 0. Then

$$\begin{aligned} E[T_A] &= \frac{1}{4} + \frac{1}{2}E[T_{AB}], \\ E[T_B] &= \frac{1}{3} + \frac{2}{3}E[T_{AB}], \\ E[T_{AB}] &= \frac{1}{3} + \frac{1}{3}E[T_A] + \frac{2}{3}E[T_B]. \end{aligned}$$

Solving these equations gives

$$E[T_A] = \frac{15}{14} \text{ (min)}.$$

c. [2 pt.] The balance equations are:

$$\begin{aligned} 2p_0 &= 2p_A + p_B, \\ 4p_A &= p_0 + p_{AB}, \\ 3p_B &= p_0 + 2p_{AB}, \\ 3p_{AB} &= 2p_A + 2p_B, \end{aligned}$$

from which, together with the normalization equation $p_0 + p_A + p_B + p_{AB} = 1$, follows that

$$p_0 = p_A = p_{AB} = \frac{2}{7}, \quad p_A = \frac{1}{7}.$$

d. [1 pt.] The fraction of calls that is lost is $P_{AB} = \frac{2}{7}$.
e. [2 pt.] Let state 1 denote the state '1 call on hold'. Then the balance equations become

$$\begin{aligned} 5p_{AB} &= p_A + p_B + 3p_1, \\ 3p_1 &= 2p_{AB}, \end{aligned}$$

while the balance equations in states 0, A and B remain unaltered. Solution gives

$$p_1 = \frac{4}{25},$$

which is the fraction of calls that is lost.

EXERCISE 5
a. [2 pt.] Let p_i denote the probability that i orders are processed during a week. Then

$$p_i = e^{-5}\frac{5^i}{i!}, \quad i = 0, 1, 2, 3, 4; \quad p_5 = 1 - \sum_{i=0}^{4} p_i = 1 - \sum_{i=0}^{4} e^{-5}\frac{5^i}{i!}.$$

b. [2 pt.] The mean number per hour is the mean number per week divided by the duration of a week, so

$$\frac{1}{40}\sum_{i=0}^{5} ip_i = \frac{1}{8}\left(1 - e^{-5}\frac{5^4}{4!}\right).$$

c. [2 pt.] Let T be the time to produce 5 orders, so T is Erlang-5 distributed with mean 40 hours. This means that the distribution function is given by

$$P(T \leq t) = 1 - \sum_{i=0}^{4} e^{-t/8}\frac{(t/8)^i}{i!}, \quad t \geq 0,$$

and the density is

$$f_T(t) = \frac{1}{8} e^{-t/8} \frac{(t/8)^4}{4!}.$$

Let B denote the time the facility is busy during a week and I denote the idle time, then

$$\begin{aligned}
E(B) &= \int_{t=0}^{40} t f_T(t) dt + 40P(T > 40) \\
&= 40 \int_{t=0}^{40} \frac{1}{8} e^{-t/8} \frac{(t/8)^5}{5!} dt + 40P(T > 40) \\
&= 40 \left(1 - \sum_{i=0}^{5} e^{-5} \frac{5^i}{i!} + \sum_{i=0}^{4} e^{-5} \frac{5^i}{i!} \right) \\
&= 40 \left(1 - e^{-5} \frac{5^5}{5!} \right) = 40 \left(1 - e^{-5} \frac{5^4}{4!} \right).
\end{aligned}$$

Hence, for the mean idle time per week we find

$$E(I) = 40 - E(B) = 40 e^{-5} \frac{5^4}{4!} \text{ (hours)}.$$

d. [2 pt.] The long-run average cost per hour is the average cost per week divided by the duration of a week, so

$$\frac{1}{40} \left(50E(I) + 100 \left[5 - \sum_{i=0}^{5} i p_i \right] \right) = \frac{250}{4} e^{-5} \frac{5^4}{4!}.$$