LNMB EXAM Introduction to Stochastic Processes (ISP)
Friday 19 November, 2010, 13.00-16.00 hours.

EXERCISE 1

a. [2 pt.] The classes of communicating states are {3,5} and {2} and {1,4,6}.
b. [2 pt.] The period of each state is 1.
c. [2 pt.] For initial states 3 and 5 the limiting distribution is

$$\pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6) = \left(0, 0, \frac{2}{5}, 0, \frac{3}{5}, 0, \right).$$

For initial states 1, 4 or 6 the limiting distribution is

$$\pi = \left(\frac{2}{3}, 0, 0, \frac{1}{6}, 0, \frac{1}{6}\right).$$

Finally, state 2 is transient and with probability $\frac{3}{8}$ the class $\{1, 4, 6\}$ is ever reached, with probability $\frac{5}{8}$ the class {3,5} is ever reached. It follows that the limiting distribution is given by

$$\pi = \left(\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{16}, \frac{3}{8}, \frac{1}{16}\right).$$

d. [2 pt.] For initial state 2, the probability that state 3 is ever reached is equal to $\frac{5}{8}$.
e. [2 pt.] The proportion of time spent in state 1 is $\frac{2}{3}$.

EXERCISE 2

a. [2 pt.] The probability that at least 5 cars have arrived in $[0, t]$ is equal to

$$1 - \sum_{n=0}^{4} e^{-p_1 \lambda t} \frac{(p_1 \lambda t)^n}{n!}.$$

b. [2 pt.] Same answer as a., since the arrivals at P_1 and P_2 are independent.
c. [2 pt.] The probability that exactly 5 cars have arrived in P_1 before the first one in P_2 is given by

$$p_1^5(1 - p_1).$$

d. [3 pt.] The 5 cars arrived uniformly in [0,10]. So the probability that a car is still present at time 10 is $\frac{7}{10}$. Hence, the probability that 3 of them are present at time 10 is

$$\binom{5}{3} \left(\frac{7}{10}\right)^3 \left(\frac{3}{10}\right)^2.$$

EXERCISE 3

a. [2 pt.] $E[Z] = \frac{2p}{1-p}$, $\text{var}[Z] = \frac{2p}{(1-p)^2}$ and $E[X_i] = E[Z]^i = \left(\frac{2p}{1-p}\right)^i$.

b. [2 pt.] The extinction probability $\pi_0 = 1$ for $p = \frac{1}{2}$. If $p = \frac{1}{2}$, then π_0 is the unique root on $(0, 1)$ of the equation

$$\pi_0 = G(\pi_0).$$

This yields

$$\pi_0 = \frac{1}{(2 - \pi_0)^2},$$

which can be reduced to

$$(\pi_0 - 1)(\pi_0^2 - 3\pi_0 + 1) = 0.$$

The root on $(0, 1)$ is given by

$$\pi_0 = \frac{3 - \sqrt{5}}{2} \, .$$

EXERCISE 4

a. [1 pt.] The state space is $\{0, 1, 2, 3\}$. The transition rate from state $n > 0$ to $n - 1$ is $2n$, and the transition rate from $n < 3$ to $n + 1$ is 6 if $n = 0$ or 1, and 3 if $n = 2$.

b. [2 pt.] Let T_n denote the time, starting from state $n > 1$, to reach 1. Then

$$\begin{aligned} E[T_3] &= \frac{1}{6} + E[T_2], \\ E[T_2] &= \frac{1}{7} + \frac{3}{7} E[T_3]. \end{aligned}$$

Solving these equations gives

$$E[T_3] = \frac{13}{24}.$$

c. [2 pt.] The equilibrium distribution is

$$\pi = (\pi_0, \pi_1, \pi_2, \pi_3) = \left(\frac{4}{43}, \frac{12}{43}, \frac{18}{43}, \frac{9}{43}\right).$$

Hence, the average number of machines in use is $\pi_1 + 2\pi_2 + 3\pi_3 = 1\frac{32}{43}$.

d. [2 pt.] The proportion of time a repairman is busy is $\pi_0 + \pi_1 + \frac{1}{2}\pi_2 = \frac{25}{43}$.

EXERCISE 5

a. [2 pt.] The proportion of *red*-events is $1 - e^{-\lambda}$.

b. [2 pt.] *Red*-events occur at rate $(1 - e^{-\lambda})\lambda$.

EXERCISE 6

a. [1 pt.] The long-run average cost are $2 \cdot 4 = 8$.

b. [2 pt.] A cycle C is the time elapsing till the part breaks down or reaches the age T (and is replaced). Then

$$E[C] = \int_0^T t\mathrm{d}t + T(1-T) = T - \frac{1}{2}T^2.$$

Let K denote the cost in a cycle, then

$$E[K] = 4 \cdot T + 1 \cdot (1-T) = 1 + 3T.$$

Hence, the long-run average cost are

$$\frac{E[K]}{E[C]} = \frac{2+6T}{2T-T^2} \ .$$

c. [3 pt.] Minimizing the average cost over T yields $T = \frac{\sqrt{7}-1}{3}$.