LNMB EXAM Introduction to Stochastic Processes (ISP)
Monday September 27, 2010, 13.15-16.15 hours.

EXERCISE 1

a. [2 pt.] The classes of communicating states are {3,4} en {6} en {1,2,5}.

b. [3 pt.] For initial states 3 and 4 the limiting distribution is

$$\pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6) = \left(0, 0, \frac{2}{5}, \frac{3}{5}, 0, 0\right).$$

For initial states 1, 2 or 5 the limiting distribution is

$$\pi = \left(\frac{15}{38}, \frac{11}{38}, 0, 0, \frac{12}{38}\right).$$

Finally, state 6 is transient and with probability $\frac{2}{3}$ the class $\{3, 4\}$ is ever reached, with probability $\frac{1}{3}$ the class {1,2,5} is ever reached. It follows that the limiting distribution is given by

$$\pi = \left(\frac{5}{38}, \frac{11}{114}, \frac{4}{15}, \frac{2}{5}, \frac{4}{38}\right).$$

c. [2 pt.] For initial state 6, the probability that state 2 is ever reached is equal to $\frac{1}{3}$.

d. [2 pt.] The mean time between visits to state 2 is $\frac{38}{11}$.

EXERCISE 2

a. [2 pt.] The probability that at least 3 persons have arrived in $[0, t]$ is equal to

$$1 - \sum_{n=0}^{2} e^{-(\lambda_c+\lambda_r)t} \frac{((\lambda_c + \lambda_r)t)^n}{n!} .$$

b. [2 pt.] The probability distribution $F(t)$ of the time until the third arrival is the same as in a., namely an Erlang-3 distribution with parameter $\lambda_c + \lambda_r$,

$$F(t) = 1 - \sum_{n=0}^{2} e^{-(\lambda_c+\lambda_r)t} \frac{((\lambda_c + \lambda_r)t)^n}{n!} .$$

c. [2 pt.] The probability that exactly 100 customers have arrived before the first bank robber is given by

$$\left(\frac{\lambda_c}{\lambda_c + \lambda_r}\right)^{100} \frac{\lambda_r}{\lambda_c + \lambda_r} .$$

d. [3 pt.] She arrived uniformly in [0,1]. So the probability that she is still present at time 1 is

$$\int_0^1 e^{-(1-y)} \mathrm{d}y = 1 - \frac{1}{e}.$$

EXERCISE 3
a. [2 pt.] $E[Z] = \frac{2(1-p)}{p}$ and $E[X_i] = E[Z]^i = \left(\frac{2(1-p)}{p}\right)^i$.
b. [2 pt.] The extinction probability $\pi_0 = 1$ for all p for which $E[Z] \leq 1$, and thus for all $\frac{2}{3} \leq p < 1$. If $0 < p < \frac{2}{3}$, then π_0 is the unique root on $(0, 1)$ of the equation

$$\pi_0 = G(\pi_0),$$

where

$$G(z) = E(z^Z) = \frac{p^2}{(1-(1-p)z)^2} \ .$$

This yields

$$\pi_0 = \frac{p^2}{(1-(1-p)\pi_0)^2},$$

which can be reduced to

$$(\pi_0 - 1)((1-p)^2\pi_0^2 - (1-p^2)\pi_0 + p^2) = 0.$$

The root on $(0, 1)$ is given by

$$\pi_0 = \frac{1 + p - \sqrt{(1+p)^2 - 4p^2}}{2(1-p)} \ .$$

EXERCISE 4
Take as 1 minute as time unit.
a. [1 pt.] The state space is $\{0, 1, 2, \ldots\}$. The transition rate from state $n \geq 0$ to $n + 1$ is 2 (per minute) and from n to $n - 1$ it is 4 if $n \geq 2$ and 2 if $n = 1$.
b. [2 pt.] Let T_n denote the time, starting from state $n < 2$, to reach 2. Then

$$\begin{aligned} E[T_0] &= \frac{1}{2} + E[T_1], \\ E[T_1] &= \frac{1}{4} + \frac{1}{2}E[T_0]. \end{aligned}$$

Solving these equations gives

$$E[T_1] = 1\,(\text{min}).$$

c. [2 pt.] The balance equations (between states $n - 1$ and n) are:

$$\begin{aligned} 2p_0 &= 2p_1, \\ 2p_{n-1} &= 4p_n, \quad n = 2, 3, \ldots, \end{aligned}$$

from which, together with the normalization equation $p_0 + p_1 + \cdots = 1$, follows that

$$p_0 = \frac{1}{3}, \quad p_n = \frac{1}{3}\left(\frac{1}{2}\right)^{n-1}, \quad n = 1, 2, \ldots.$$

d. [2 pt.] The average length of a quiet period is $E[T_1] = 1$ (min).
e. [2 pt.] Denote by C the length of a crowded period. Then we have

$$\frac{E[C]}{E[T_1] + E[C]} = p_2 + p_3 + \cdots = \frac{1}{3}.$$

Hence, $E[C] = \frac{1}{2}$ (min).

EXERCISE 5
a. [3 pt.] Let p_n denote the long-run fraction of time that the population consists of n members. The balance equations state

$$p_n(\lambda + \mu) = p_{n-1}\lambda, \quad n = 1, 2, \ldots$$

Hence,

$$p_n = \left(\frac{\lambda}{\lambda + \mu}\right)^n p_0, \quad n = 1, 2, \ldots$$

and p_0 follows from the normalization equation $p_0 + p_1 + \cdots = 1$, yielding

$$p_0 = \frac{\mu}{\lambda + \mu}.$$

Alternatively, define a cycle C as the time between two subsequent disasters. The expected cycle length is equal to $E(C) = \frac{1}{\mu}$. The expected time that the population is of size n in a cycle is

$$P(\#\text{ reaches } n) \cdot E(\text{Time } \# \text{ is } n | \# \text{ reaches } n) = \left(\frac{\lambda}{\lambda + \mu}\right)^n \cdot \frac{1}{\lambda + \mu}.$$

Thus the long-run fraction of time that the population size is n is equal to

$$\frac{\left(\frac{\lambda}{\lambda+\mu}\right)^n \frac{1}{\lambda+\mu}}{\frac{1}{\mu}} = \left(\frac{\lambda}{\lambda + \mu}\right)^n \frac{\mu}{\lambda + \mu}.$$

b. [2 pt.] The mean population size just before a disaster is $\lambda \cdot \frac{1}{\mu} = \frac{\lambda}{\mu}$.
c. [2 pt.] The long-run average population size is

$$\sum_{n=0}^{\infty} n p_n = \frac{\lambda}{\mu}.$$

Alternatively, suppose that a member costs 1 unit per time unit while being alive. Then the average population is equal to the average cost per time unit. Since each member has an exponential life with mean $\frac{1}{\mu}$ (memoryless!) and λ members are generated per time unit, it follows that the average cost per time unit are equal to $\lambda \cdot \frac{1}{\mu} = \frac{\lambda}{\mu}$. As second alternative, note that the expected cost in a cycle of length x is given by $\lambda x \cdot \frac{1}{2}x$ (since the mean number of arrivals is λx, and the average lifetime is $\frac{1}{2}x$). So the expected cost in a cycle is

$$\int_0^\infty \frac{1}{2}\lambda x^2 \mu e^{-\mu x} dx = \frac{\lambda}{\mu^2} .$$

Dividing these expected cost by the expected cycle length gives the average cost per time unit, $\frac{\lambda}{\mu}$.

d. [2 pt.] The same answer as in a., namely the probability that the population size is n just before a disaster is equal to

$$\left(\frac{\lambda}{\lambda+\mu}\right)^n \frac{\mu}{\lambda+\mu} .$$