1.

a. The state space is $\{0, 1, \ldots, c-1, c\}$, and the birth rates are

$$\lambda_n = \lambda, \quad n = 0, 1, \ldots, c-1,$$

and the death rates

$$\mu_n = n\mu, \quad n = 1, 2, \ldots, c.$$

Hence the transition rate matrix Q is given by

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \cdots & 0 \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & (c-2)\mu & -(\lambda+(c-2)\mu) & \lambda & 0 \\ 0 & \cdots & 0 & 0 & (c-1)\mu & -(\lambda+(c-1)\mu) & \lambda \\ 0 & \cdots & 0 & 0 & 0 & c\mu & -c\mu \end{pmatrix}$$

b. Now the state space does not stop at c, i.e., it is $\{0, 1, \ldots\}$ and the birth rates are

$$\lambda_n = \lambda, \quad n = 0, 1, \ldots,$$

and the death rates

$$\mu_n = n\mu, \quad n = 1, 2, \ldots.$$

2.

a. The forward equations are

$$\begin{aligned} P_{00}'(t) &= \mu P_{01}(t) - \lambda P_{00}(t) \\ P_{01}'(t) &= \lambda P_{00}(t) - \mu P_{01}(t). \end{aligned}$$

Using $P_{01}(t) = 1 - P_{00}(t)$, we obtain the following differential equation,

$$P_{00}'(t) = \mu - (\mu + \lambda) P_{00}(t).$$

b. The backward equations are

$$\begin{aligned} P_{11}'(t) &= \mu P_{01}(t) - \mu P_{11}(t) \\ P_{01}'(t) &= \lambda P_{11}(t) - \lambda P_{01}(t). \end{aligned}$$

Multiplying these equations by λ and μ, respectively, and then adding them, we obtain

$$\lambda P_{11}'(t) + \mu P_{01}'(t) = 0.$$

Hence, for some c,

$$\lambda P_{11}(t) + \mu P_{01}(t) = c.$$

Substituting $t = 0$ yields $c = \lambda$. Using the above equation, we get

$$P_{11}'(t) = \lambda - (\lambda + \mu) P_{11}(t).$$

The forward equations are

$$\begin{aligned} P_{11}'(t) &= \lambda P_{10}(t) - \mu P_{11}(t) \\ P_{10}'(t) &= \mu P_{11}(t) - \lambda P_{10}(t). \end{aligned}$$

By substituting $P_{10}(t) = 1 - P_{11}(t)$, we obtain the same differential equation as before.

c. Solving the differential equation for $P_{00}(t)$ with initial condition $P_{00}(0) = 1$ yields

$$P_{00}(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}, \quad t \geq 0.$$

3. The number of failed machines is a birth and death process with

$$\lambda_0 = 2\lambda, \quad \lambda_1 = \lambda, \quad \lambda_n = 0, n > 1, \quad \mu_1 = \mu_2 = \mu, \quad \mu_n = 0, n \neq 1, 2.$$

Now substitute into the backward equations.

4. Let

$$I_j(t) = \begin{cases} 0, & \text{if machine } j \text{ is working at time } t, \\ 1, & \text{otherwise.} \end{cases}$$

Also, let the state be $(I_1(t), I_2(t))$. This is clearly a continuous-time Markov chain with

$$\begin{aligned}
v_{(0,0)} &= \lambda_1 + \lambda_2, \quad \lambda_{(0,0);(0,1)} = \lambda_2, \quad \lambda_{(0,0);(1,0)} = \lambda_1, \\
v_{(0,1)} &= \lambda_1 + \mu_2, \quad \lambda_{(0,1);(0,0)} = \mu_2, \quad \lambda_{(0,1);(1,1)} = \lambda_1, \\
v_{(1,0)} &= \mu_1 + \lambda_2, \quad \lambda_{(1,0);(0,0)} = \mu_1, \quad \lambda_{(1,0);(1,1)} = \lambda_2, \\
v_{(1,1)} &= \mu_1 + \mu_2, \quad \lambda_{(1,1);(0,1)} = \mu_1, \quad \lambda_{(1,1);(1,0)} = \mu_2.
\end{aligned}$$

By the independence assumption we have

$$P_{(i,j),(k,l)}(t) = P_{i,k}(t) Q_{j,l}(t), \tag{1}$$

where $P_{i,k}(t)$ is the probability that the first machine is in state k at time t given that it was at state i at time 0; $Q_{j,l}(t)$ is defined similarly for the second machine. By example 4.11 we have

$$\begin{aligned}
P_{0,0}(t) &= \frac{\mu_1}{\lambda_1 + \mu_1} + \frac{\lambda_1}{\lambda_1 + \mu_1} e^{-(\lambda_1+\mu_1)t}, \\
P_{1,0}(t) &= \frac{\mu_1}{\lambda_1 + \mu_1} - \frac{\mu_1}{\lambda_1 + \mu_1} e^{-(\lambda_1+\mu_1)t},
\end{aligned}$$

and by the same argument,

$$\begin{aligned}P_{1,1}(t) &= \frac{\lambda_1}{\lambda_1+\mu_1} + \frac{\mu_1}{\lambda_1+\mu_1} e^{-(\lambda_1+\mu_1)t},\\ P_{0,1}(t) &= \frac{\lambda_1}{\lambda_1+\mu_1} - \frac{\lambda_1}{\lambda_1+\mu_1} e^{-(\lambda_1+\mu_1)t}.\end{aligned}$$

Of course, similar expressions for the second machine are obtained by replacing (λ_1, μ_1) by (λ_2, μ_2). We then get $P_{(i,j),(k,l)}(t)$ by formula (**??**). For instance,

$$P_{(0,0),(0,0)}(t) = P_{0,0}(t)Q_{0,0}(t) = \frac{\mu_1 + \lambda_1 e^{-(\lambda_1+\mu_1)t}}{\lambda_1+\mu_1} \cdot \frac{\mu_2 + \lambda_2 e^{-(\lambda_2+\mu_2)t}}{\lambda_2+\mu_2}.$$