1.

a. If the state is the number of individuals at time t, we get a birth and death processs with

$$\begin{aligned} \lambda_n &= n\lambda + \theta, \qquad n < N, \\ \lambda_n &= n\lambda \qquad n \geq N, \\ \mu_n &= n\mu. \end{aligned}$$

b. Let P_i be the long-run probability that the system is in state i. Since this is also the proportion of time the system is in state i, we are looking for

$$\sum_{i=3}^{\infty} P_i.$$

We have

$$P_k\mu_k = P_{k-1}\lambda_{k-1}, \qquad k = 1, 2, \ldots.$$

This yields

$$\begin{aligned} P_1 &= \frac{\theta}{\mu} P_0 = \frac{1}{2} P_0, \\ P_2 &= \frac{\theta+\lambda}{2\mu} P_1 = \frac{1}{2} P_1 = \frac{1}{4} P_0, \\ P_3 &= \frac{\theta+2\lambda}{3\mu} P_2 = \frac{1}{2} P_2 = \frac{1}{8} P_0, \end{aligned}$$

and for $k \geq 3$,

$$P_k = \frac{(k-1)\lambda}{k\mu} P_{k-1} = \frac{k-1}{k}\frac{1}{2} P_{k-1} = \cdots = \frac{3}{k}\left(\frac{1}{2}\right)^{k-3} P_3.$$

Hence

$$\sum_{k=3}^{\infty} P_k = P_3 \sum_{k=3}^{\infty} \frac{3}{k}\left(\frac{1}{2}\right)^{k-3} = 24 P_3 \sum_{k=3}^{\infty} \frac{1}{k}\left(\frac{1}{2}\right)^{k}.$$

Since

$$\sum_{k=1}^{\infty} \frac{1}{k} x^k = -\log(1-x),$$

we get

$$\sum_{k=3}^{\infty} P_k = P_3(24\log 2 - 15).$$

Because all probabilities add up to 1, we have

$$P_0\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}(24\log 2 - 15)\right) = 1.$$

So

$$P_0^{-1} = 3\log 2 - \frac{1}{8},$$

and thus finally,

$$\sum_{k=3}^{\infty} P_k = \frac{24\log 2 - 15}{24\log 2 - 1} \approx 0.105.$$

2. With the number of customers in the shop as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = 3, \qquad \mu_1 = \mu_2 = 4.$$

Therefore

$$P_1 = \frac{3}{4}P_0, \qquad P_2 = \frac{3}{4}P_1 = \left(\frac{3}{4}\right)^2 P_0.$$

And since $P_0 + P_1 + P_2 = 1$, we get

$$P_0 = \frac{16}{37}.$$

a. The average number of customers in the shop is

$$P_1 + 2P_2 = \frac{30}{37}.$$

b. The proportion of the customers that enter the shop is

$$\frac{\lambda(1 - P_2)}{\lambda} = 1 - P_2 = \frac{28}{37}.$$

c. Now $\mu = 8$, so

$$P_0 = \frac{64}{97}.$$

So the proportion of the customers that enter the shop is

$$1 - P_2 = \frac{88}{97}.$$

The rate of added customers is therefore

$$\lambda\left(\frac{88}{97}\right) - \lambda\left(\frac{28}{37}\right) \approx 0.45.$$

The business he does would improve by 0.45 customers per hour.

3. With the number of customers in the system as the state, we get a birth and death process with

$$\lambda_0 = \lambda_1 = \lambda_2 = 3, \quad \lambda_i = 0, i \geq 4, \qquad \mu_1 = 2, \mu_2 = \mu_3 = 4.$$

Therefore the balance equations reduce to

$$P_1 = \frac{3}{2}P_0, \quad P_2 = \frac{3}{4}P_1 = \frac{9}{8}P_0, \quad P_3 = \frac{3}{4}P_2 = \frac{27}{32}P_0.$$

And therefore,

$$P_0 = \left(1 + \frac{3}{2} + \frac{9}{8} + \frac{27}{32}\right)^{-1} = \frac{32}{143}.$$

a. The fraction of potential custoemrs that enter the system is

$$\frac{\lambda(1-P_3)}{\lambda} = 1 - P_3 = \frac{116}{143}.$$

b. With a server working twice as fast we would get

$$P_1 = \frac{3}{4}P_0, \quad P_2 = \frac{3}{4}P_1 = \left(\frac{3}{4}\right)^2 P_0, \quad P_3 = \left(\frac{3}{4}\right)^3 P_0,$$

and

$$P_0 = \left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3\right)^{-1} = \frac{64}{175}.$$

So that now

$$1 - P_3 = \frac{148}{175}.$$

4. Say the state is 0 if the machine is up, say it is in state i when it is down due to a type i failure, $i = 1, 2$. The balance equations for the limiting probabilities are as follows.

$$\begin{aligned} \lambda P_0 &= \mu_1 P_1 + \mu_2 P_2, \\ \mu_1 P_1 &= \lambda p P_0, \\ \mu_2 P_2 &= \lambda(1-p)P_0, \\ P_0 + P_1 + P_2 &= 1. \end{aligned}$$

These equations are easily solved to give the results

$$P_0 = (1 + \lambda p/\mu_1 + \lambda(1-p)/\mu_2)^{-1}, \quad P_1 = \lambda p P_0/\mu_1, \quad P_2 = \lambda(1-p)P_0/\mu_2.$$