1. There are $k+1$ states: state 0 means the machine is working, state i means that it is in repair phase i, $i = 1, \dots, k$. The balance equations for the limiting probabilities are

$$\begin{aligned}
\lambda P_0 &= \mu_k P_k, \\
\mu_1 P_1 &= \lambda P_0, \\
\mu_i P_i &= \mu_{i-1} P_{i-1}, \quad i = 2, \dots, k,
\end{aligned}$$

and the normalization equation is

$$P_0 + \cdots + P_k = 1.$$

To solve, note that

$$\mu_i P_i = \mu_{i-1} P_{i-1} = \mu_{i-2} P_{i-2} = \cdots = \lambda P_0.$$

Hence,

$$P_i = (\lambda/\mu_i) P_0,$$

and, upon summing,

$$1 = P_0 \left(1 + \sum_{i=1}^{k} \frac{\lambda}{\mu_i}\right).$$

Therefore,

$$P_0 = \left(1 + \sum_{i=1}^{k} \frac{\lambda}{\mu_i}\right)^{-1}, \qquad P_i = (\lambda/\mu_i) P_0, \quad i = 1, \dots, k.$$

a. P_i.

b. P_0.

2. The number in the system is a birth and death process with parameters

$$\lambda_n = \lambda/(n+1), \quad n \geq 0, \qquad \mu_n = \mu, \quad n \geq 1.$$

From the equation above (6.20),

$$1/P_0 = 1 + \sum_{n=1}^{\infty} (\lambda/\mu)^n / n! = e^{\lambda/\mu}$$

and from (6.20),

$$P_n = P_0 (\lambda/\mu)^n / n! = e^{-\lambda/\mu} (\lambda/\mu)^n / n!, \qquad n \geq 0.$$

3. Let the state denote the number of machines that are down. This yields a birth and death process with

$$\lambda_0 = \frac{3}{10}, \quad \lambda_1 = \frac{2}{10}, \quad \lambda_2 = \frac{1}{10}, \quad \lambda_i = 0, \quad i \geq 3,$$

and

$$\mu_1 = \frac{1}{8}, \quad \mu_2 = \frac{2}{8}, \quad \mu_3 = \frac{2}{8}.$$

The balance equations reduce to

$$\begin{aligned} P_1 &= \frac{3/10}{1/8} P_0 = \frac{12}{5} P_0, \\ P_2 &= \frac{2/10}{2/8} P_1 = \frac{4}{5} P_1 = \frac{48}{25} P_0, \\ P_3 &= \frac{1/10}{2/8} P_2 = \frac{4}{10} P_2 = \frac{192}{250} P_0. \end{aligned}$$

Hence, using $P_0 + P_1 + P_2 + P_3 = 1$, yields

$$P_0 = \left(1 + \frac{12}{5} + \frac{48}{25} + \frac{192}{250}\right)^{-1} = \frac{250}{1522}.$$

a. Average number not in use is

$$P_1 + 2P_2 + 3P_3 = \frac{2136}{1522} = \frac{1068}{761}.$$

b. Proportion of time both repairmen are busy is

$$P_2 + P_3 = \frac{336}{761}.$$

4.

a. $\pi_0 = \frac{1}{3}$.

b. $\pi_0 = 1$.

c. $\pi_0 = (\sqrt{3} - 1)/\sqrt{2}$.