1.

a. The extinction probability π_0 is the smallest positive solution of

$$\pi_0 = \frac{1}{5} + \frac{3}{5}\pi_0 + \frac{1}{5}\pi_0^2,$$

yielding $\pi_0 = 1$.

b. Similar as part a, but now we find $\pi_0 = \frac{1}{2}$.

c. The extinction probability π_0 is the square of the answers in part a and b.

2. Here we have

$$\mu = \frac{1-\alpha}{3}(0+2+3) + \alpha = \frac{5-2\alpha}{3}.$$

a. $E[X_n] = E[X_0]\mu^n = \left(\frac{5-2\alpha}{3}\right)^n$.

b. The extinction probability π_0 is the smallest positive solution of

$$\pi_0 = \frac{1-\alpha}{3}\left(1 + \pi_0^2 + \pi_0^3\right) + \alpha\pi_0,$$

yielding

$$\pi_0 = \sqrt{2} - 1.$$

c. Clearly,

$$E[X_0] = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = 2$$

so $E[X_n] = 2\left(\frac{5-2\alpha}{3}\right)^n$. Further,

$$\pi_0 = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n\left(\sqrt{2}-1\right)^n = \frac{\sqrt{2}-1}{3-\sqrt{2}} \quad \left(< \sqrt{2}-1\right).$$

3.

a. By conditioning on the first event and using that all particles behave independently of each otther, we get

$$\pi_0 = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\pi_0^2,$$

the roots of which are 1 and $\frac{\mu}{\lambda}$. Hence, if $\lambda \le \mu$, then $\pi_0 = 1$, and otherwise, $\pi_0 = \frac{\mu}{\lambda}$.

b. Clearly $\{X_t, t \ge 0\}$ is a birth and death process with birth rates $\lambda_n = n\lambda$ and death rates $\mu_n = n\mu$, $n = 1, 2, \ldots$ and state 0 is an absorbing state.

4.

a. S_n is Poisson with mean $n\mu$.

b.

$$\begin{aligned} P(N(t)=n) &= P(N(t)\geq n) - P(N(t)\geq n+1) \\ &= P(S_n \leq t) - P(S_{n+1}\leq t) \\ &= \sum_{k=0}^{\lfloor t \rfloor} e^{-n\mu}\frac{(n\mu)^k}{k!} - \sum_{k=0}^{\lfloor t \rfloor} e^{-(n+1)\mu}\frac{((n+1)\mu)^k}{k!}. \end{aligned}$$

5.

a. No. Suppose, for instance, that the interarrival times of the first renewal process are identically equal to 1. Let the second be a Poisson process with rate λ. If the first interarrival time of the process $\{N(t), t \geq 0\}$ is equal to 3/4, then we can be certain that the next one is less than or equal to 1/4.

b. No. Use the same processes as in a for a counterexample. For instance, the first interarrival will equal 1 with probability $e^{-\lambda}$. The probability will be different for the next interarrival.

c. No, because of a or b.