1. Since, from Example 7.3, $m(t) = e^t - 1$, $0 < t \leq 1$, we obtain upon using the identity $t + E[Y(t)] = \mu(m(t) + 1)$ that $E[Y(1)] = e/2 - 1$.

2. The long-run average cost is

$$\frac{[c + 2c + \cdots + (N-1)c]/\lambda + KNc + \lambda K^2 c/2}{N/\lambda + K} = \frac{c(N-1)N/2\lambda + KNc + \lambda K^2 c/2}{N/\lambda + K}.$$

3. Say that the system is on at t if $X_{N(t)+1}$, the interarrival time at t, is less than c (and off otherwise). Hence, the proportion of time that $X_{N(t)+1}$ is less than c is

$$\frac{E[\text{on time in a renewal cycle}]}{E[\text{cycle time}]} = \frac{\int_0^c t f(t) dt}{E[X]}.$$

4.

a. This is an alternating renewal process, with the mean off time obtained by conditioning on which machine fails to cause the off period.

$$\begin{aligned} E[\text{off}] &= \sum_{i=1}^{3} E[\text{off}|i \text{ fails}]P(i \text{ fails}) \\ &= \frac{1}{5}\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} + 2\frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{3}{2}\frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}. \end{aligned}$$

As the on time in a cycle is exponential with rate equal to $\lambda_1 + \lambda_2 + \lambda_3$, we obtain that p, the proportion of time that the system is working is

$$p = \frac{1/(\lambda_1 + \lambda_2 + \lambda_3)}{E[C]},$$

where

$$E[C] = E[\text{cycle time}] = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + E[\text{off}].$$

b. Think of the system as a renewal reward process by supposing that we earn 1 per unit of time that machine 1 is being repaired. Then, r_1, the proportion of time that machine 1 is being repaired is

$$r_1 = \frac{\frac{1}{5}\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}}{E[C]}.$$

c. By assuming that we earn 1 per unit time when machine 2 is in a state of suspended animation, shows that, with s_2 being the proportion of time that 2 is in a state of suspended animation,

$$s_2 = \frac{\frac{1}{5}\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} + \frac{3}{2}\frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}}{E[C]}.$$

5. Let T be the time it takes the shuttle to return. Now, given T, X is Poisson with mean λT. Thus,

$$E[X|T] = \lambda T, \quad \mathrm{Var}(X|T) = \lambda T.$$

Consequently,

a. $E[X] = E[E[X|T]] = \lambda E[T]$.

b. $\mathrm{Var}(X) = E[\mathrm{Var}(X|T)] + \mathrm{Var}(E[X|T])(\text{see Proposition 3.1}) = \lambda E[T] + \lambda^2 \mathrm{Var}(T)$.

c. Assume that a reward of 1 is earned each time the shuttle returns empty. Then, from the renewal reward theory, r, the rate at which the shuttle returns empty, is

$$\begin{aligned} r &= \frac{P(\text{empty})}{E[T]} \\ &= \frac{\int_0^\infty P(\text{empty}|T = t) f(t) dt}{E[T]} \\ &= \frac{\int_0^\infty e^{-\lambda t} f(t) dt}{E[T]} \\ &= \frac{E[e^{-\lambda T}]}{E[T]} . \end{aligned}$$

d. Assume that a reward of 1 is earned each time that a customer writes an angry letter. Then, with N_a equal to the number of angry letters written in a cycle, it follows that r_a, the rate at which angry letters are written, is

$$\begin{aligned} r_a &= E[N_a]/E[T] \\ &= \int_0^\infty E[N_a|T = t] f(t) dt / E[T] \\ &= \int_c^\infty \lambda(t - c) f(t) dt / E[T] \\ &= \lambda E[\max\{0, T - c\}]/E[T] . \end{aligned}$$

Since passengers arrive at rate λ, this implies that the proportion of passengers that write angry letters is r_a/λ.

e. Because passengers arrive at a constant rate, the proportion of them that have to wait more than c will equal the proportion of time that the age of the renewal process (whose event times are the return times of the shuttle) is greater than c. It is thus equal to $1 - F_e(c)$.