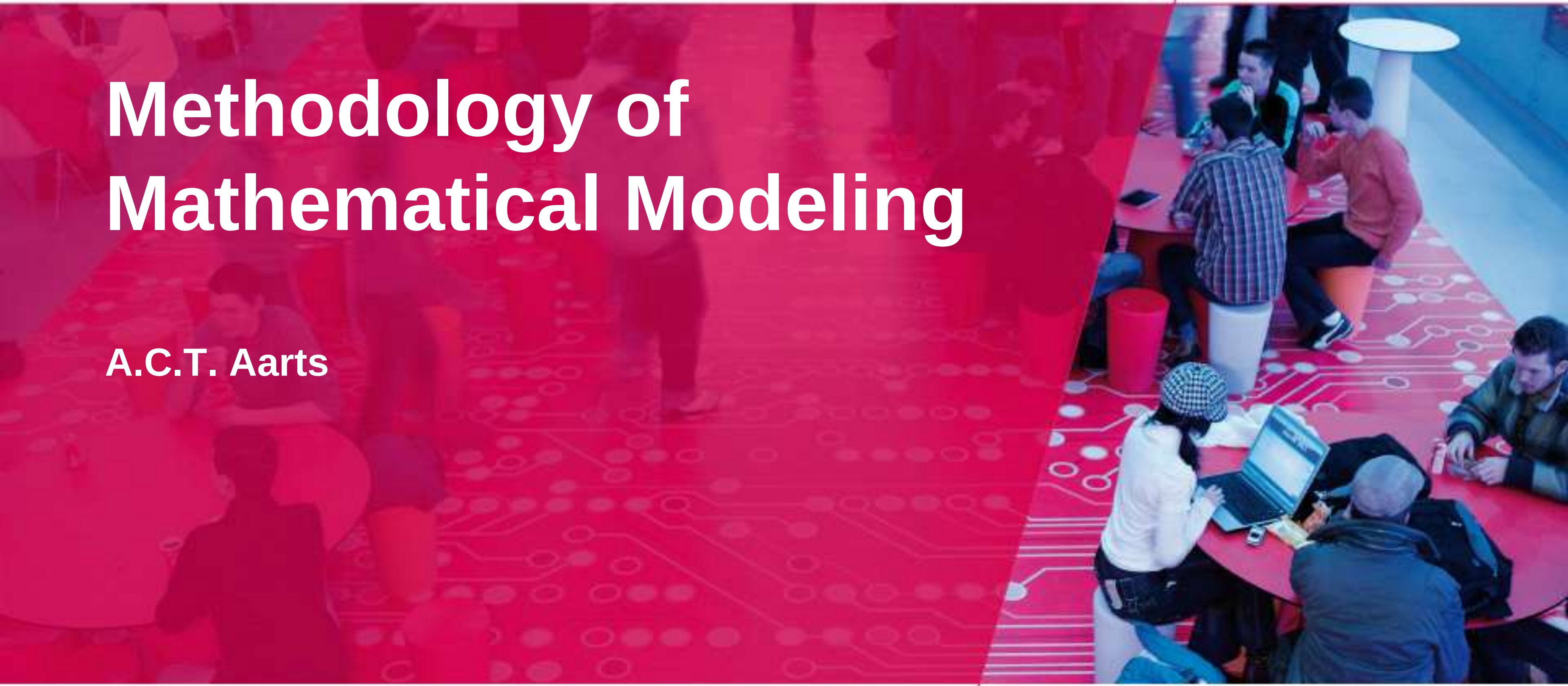

Methodology of Mathematical Modeling

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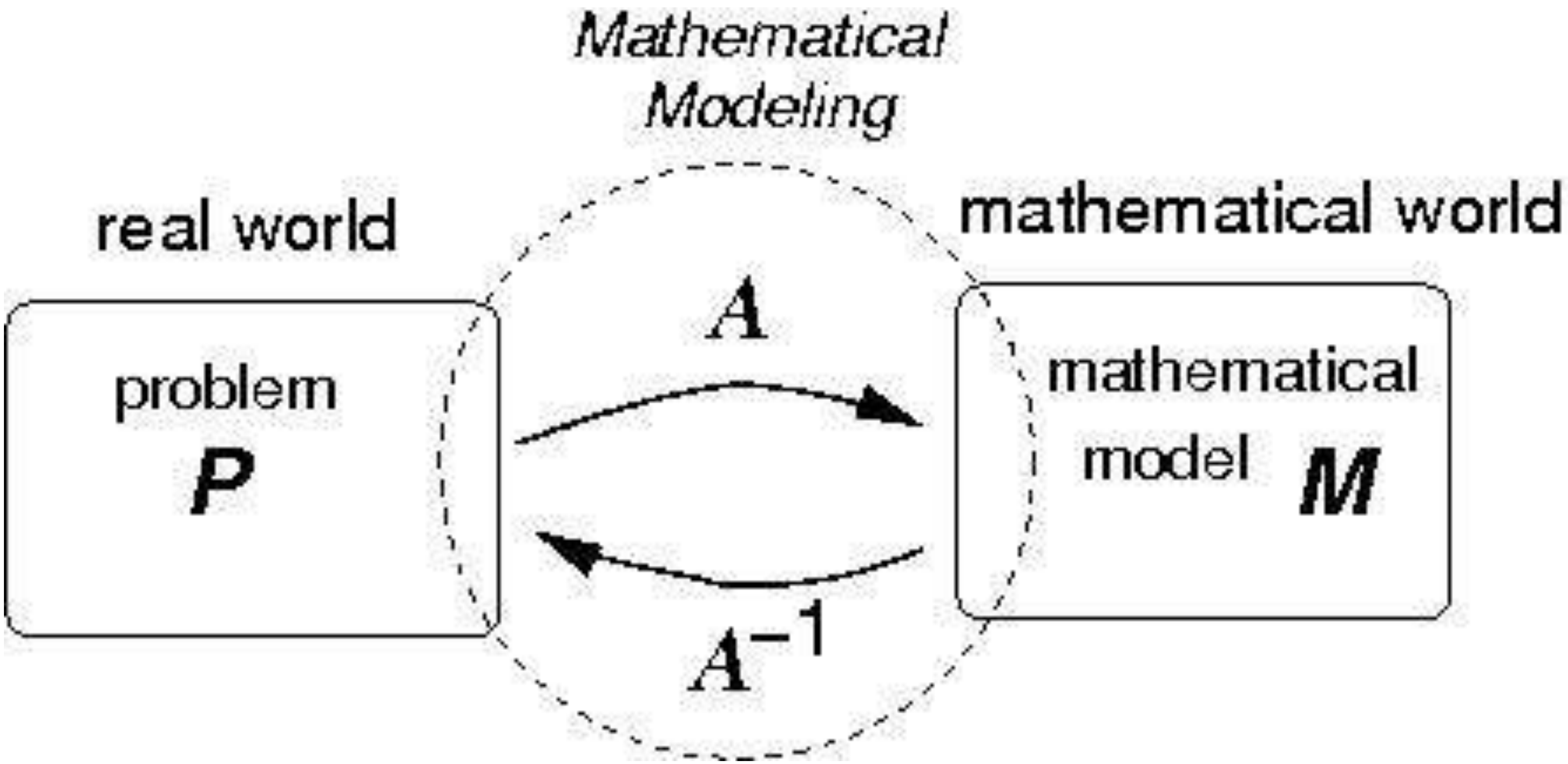
TU/e Technische Universiteit Eindhoven University of Technology

Where innovation starts

Contents

Mathematical Modeling

What is a Mathematical Modeling?



Models

- **What is a model?**

Simplified representation of certain aspects of a real system, capturing the essence of that system

- **What is a mathematical model?**

Model created using mathematical concepts (variables, operators, functions, inequalities,…)

Mathematical Models

- **First-Principle Models**

  based on physical laws (e.g. Newton's second law)

  descriptive, explaining

  material parameter values often not known (measurements)

- **Stochastic Models**

  based on distributions, averages (e.g. risk models)

  capable to deal with random phenomena**,**

  hard to distinguish relations

- **Empirical/data Models**

  based on (historical) patterns, data (e.g. Moore's law)

  not explaining, relations based on reality

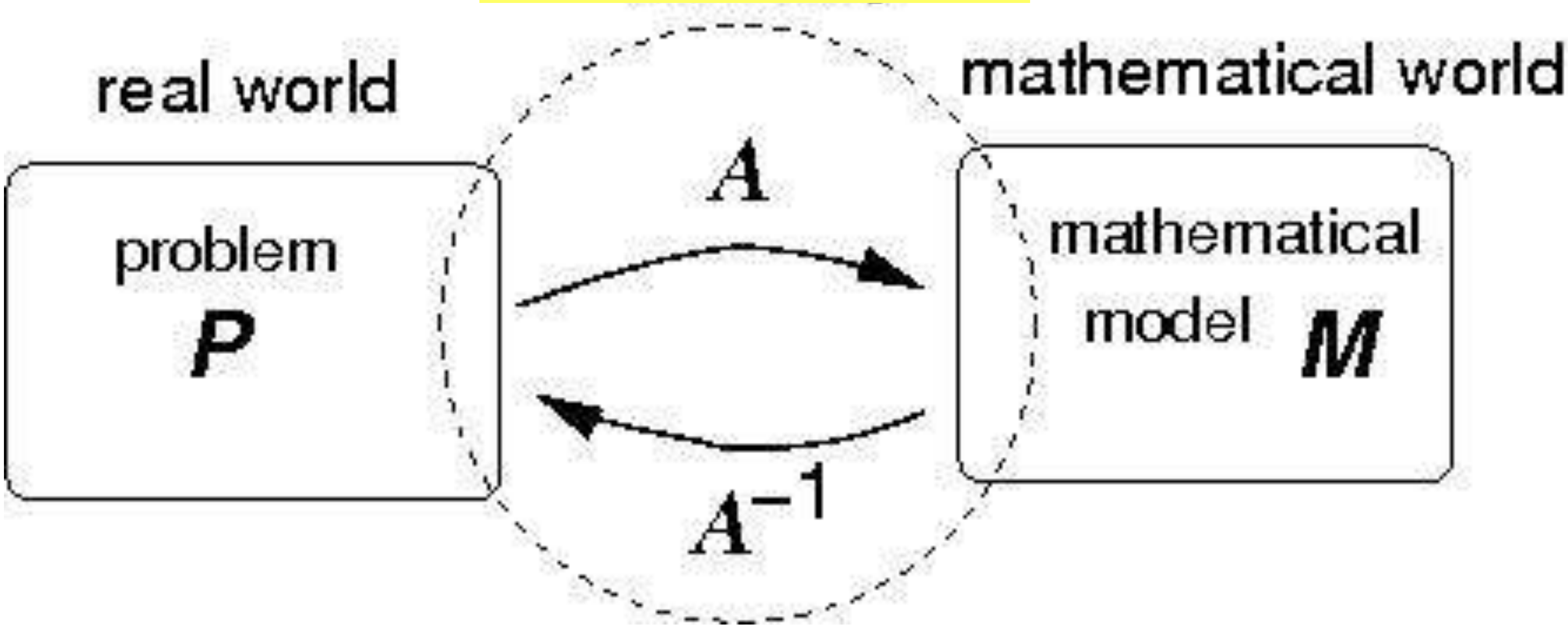
Mathematical Models

- **Descriptive vs. stochastic**
- **Continuum vs. discrete**
- **Qualitative vs. quantitative**

Mathematical Modeling

- **Mental Image**
- **Non-uniqueness**
- **People involved**
- **Outcome**
- **Skills**

Mathematical Modeling



Mental Image

- Forming of mental image of situation being modeled
- Symbolizes the problem
- Using concepts & analogies

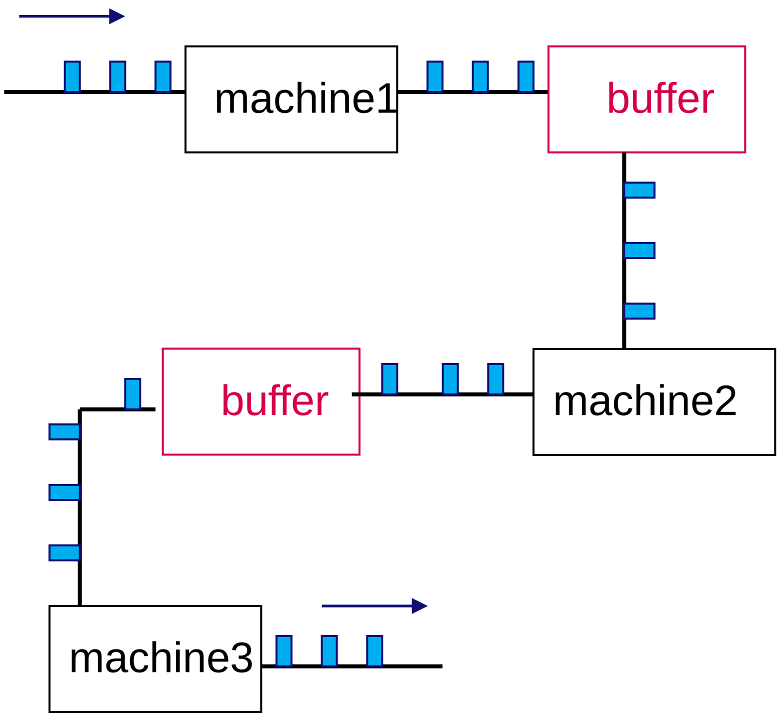
Conceptual thinking

Creativity

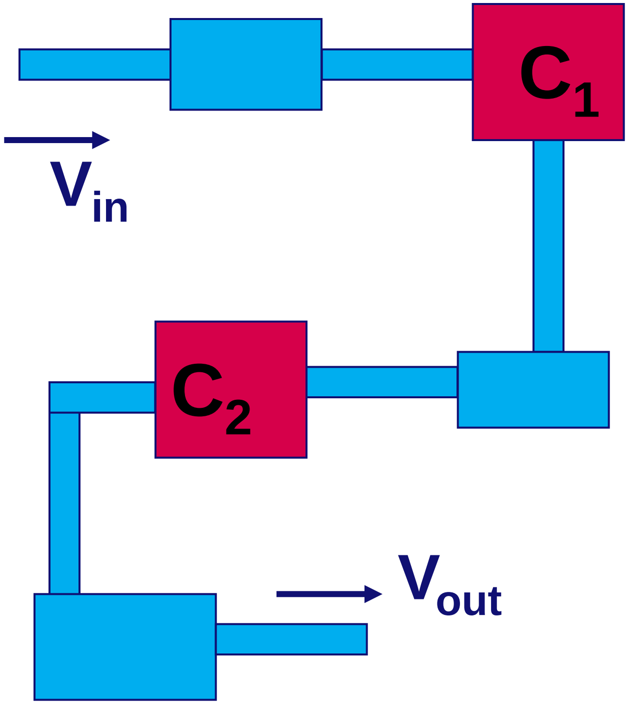
Simplification

Mental Image: example

Production line:



Fluid flow: *metaphor*

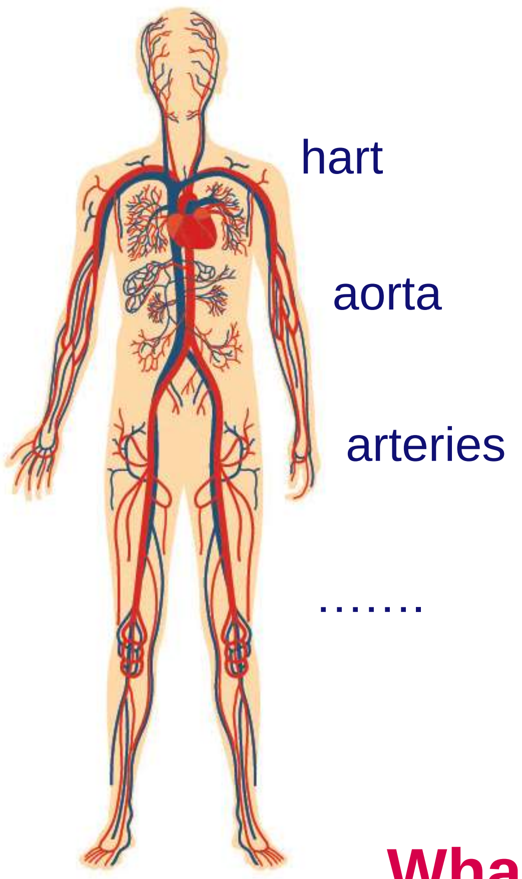


Optimise production line;

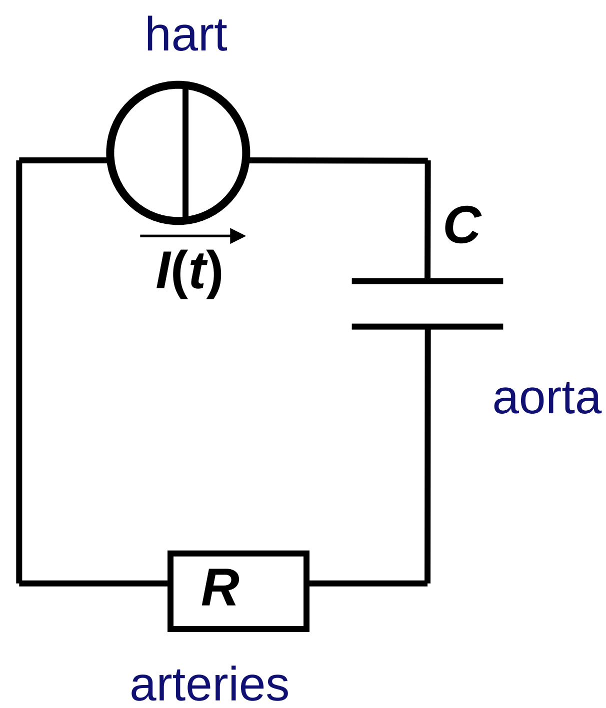
Can buffers be removed/reduced?

Mental Image: example

Blood circulation:



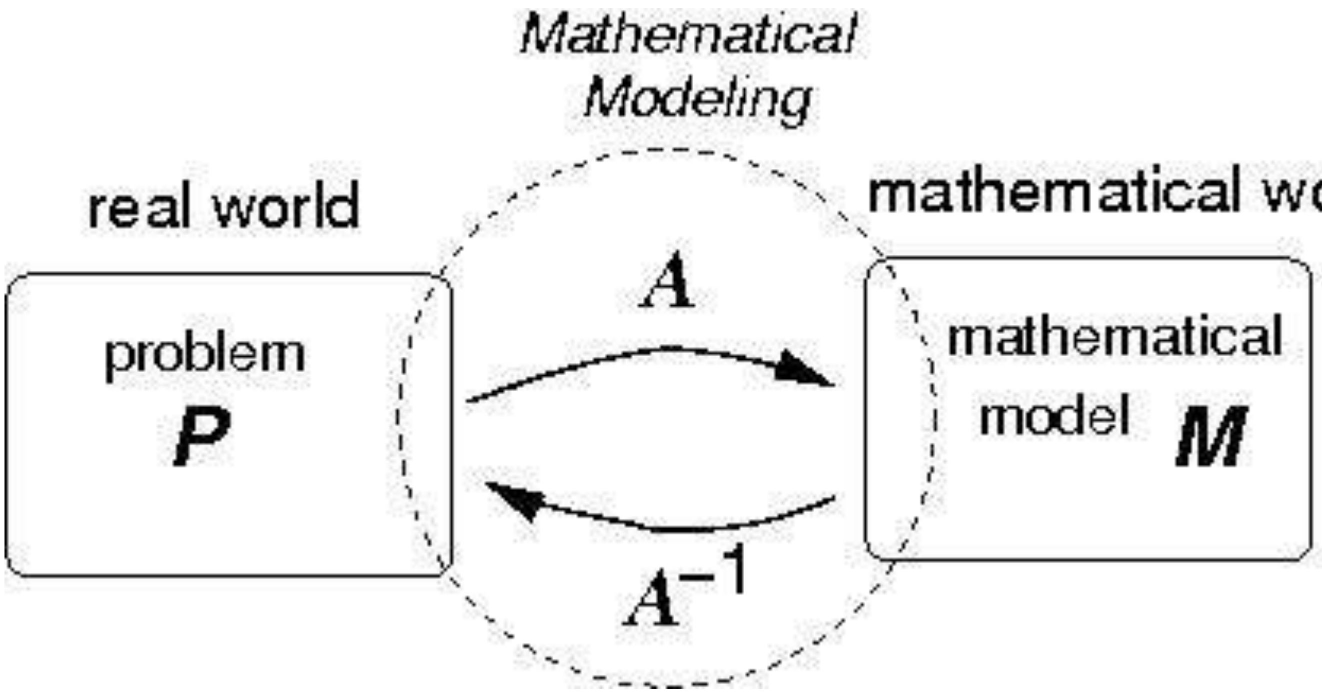
Electrical circuit:



What is the effect of age & medicine on blood pressure ?

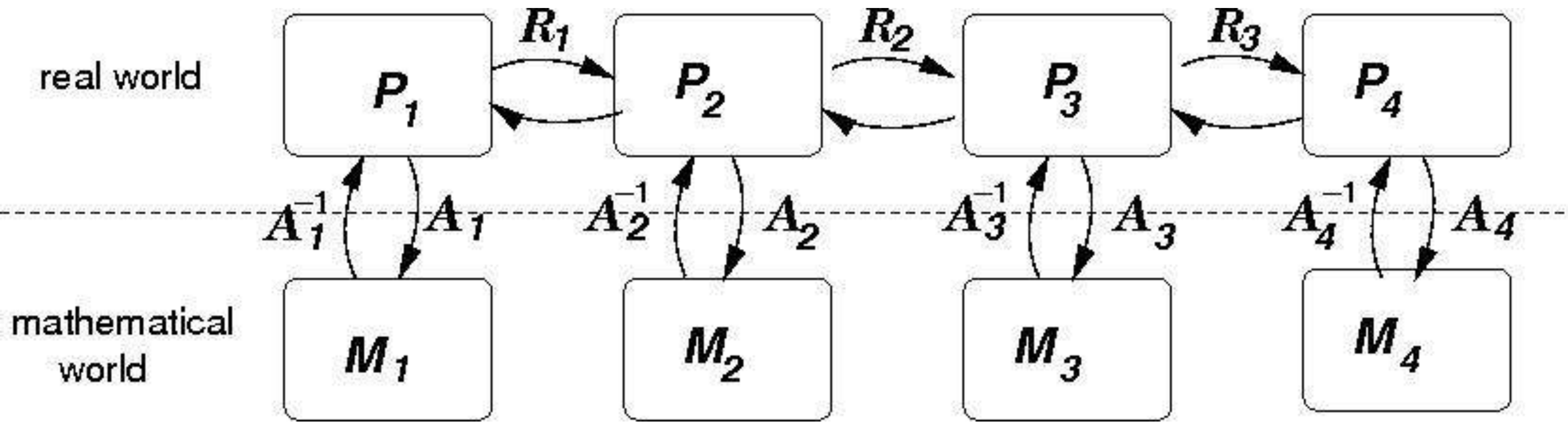
People Involved

Problem Owner | **Mathematician**



Non-uniqueness

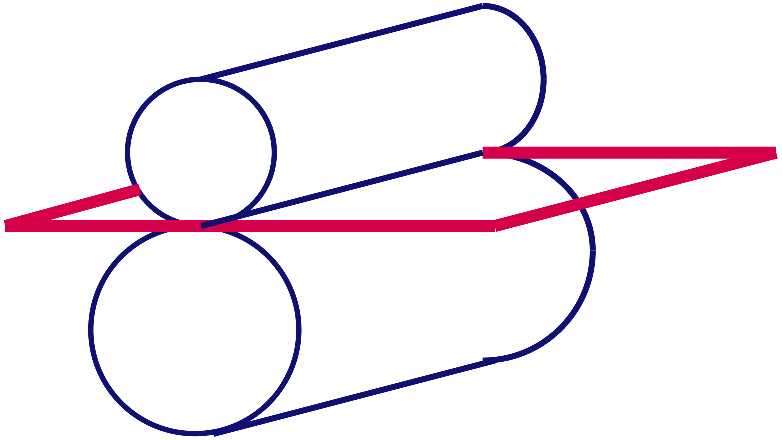
Simplification:

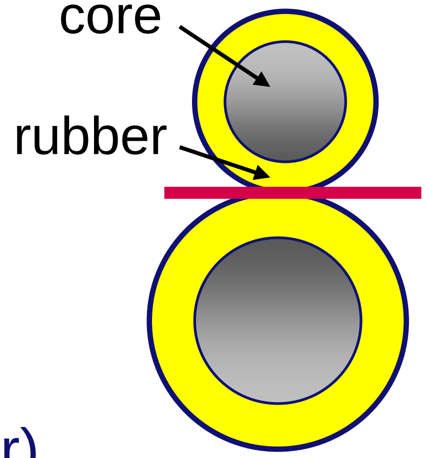


Non-uniqueness: choice of model depends on

- the form the solution needs to be (problem) accuracy
- availability of data

Simplification: example





Simplifications (in consultation with Problem Owner)

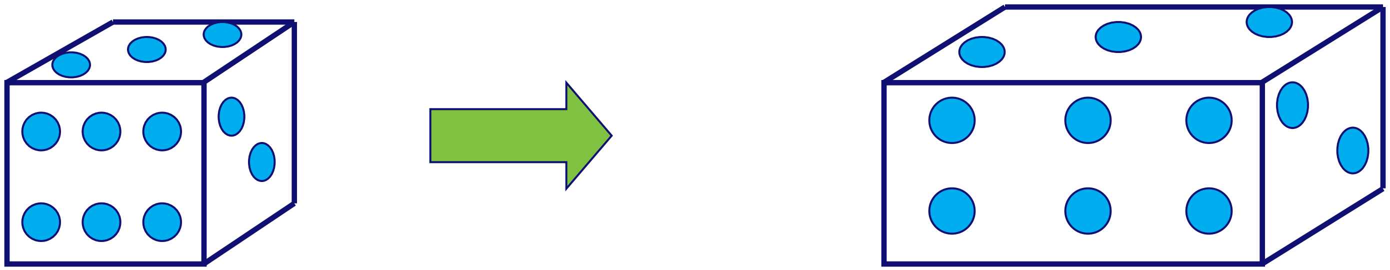
1. Core is bending:
   - 1D thin-beam bending:

$$\frac{d^2}{dz^2}\left(EI\frac{d^2u_b}{dz^2}\right)=p$$

2. Rubber layer is compressed:
   - 1D-model: empirical law:

$$u_c = ap^b$$

Non-uniqueness



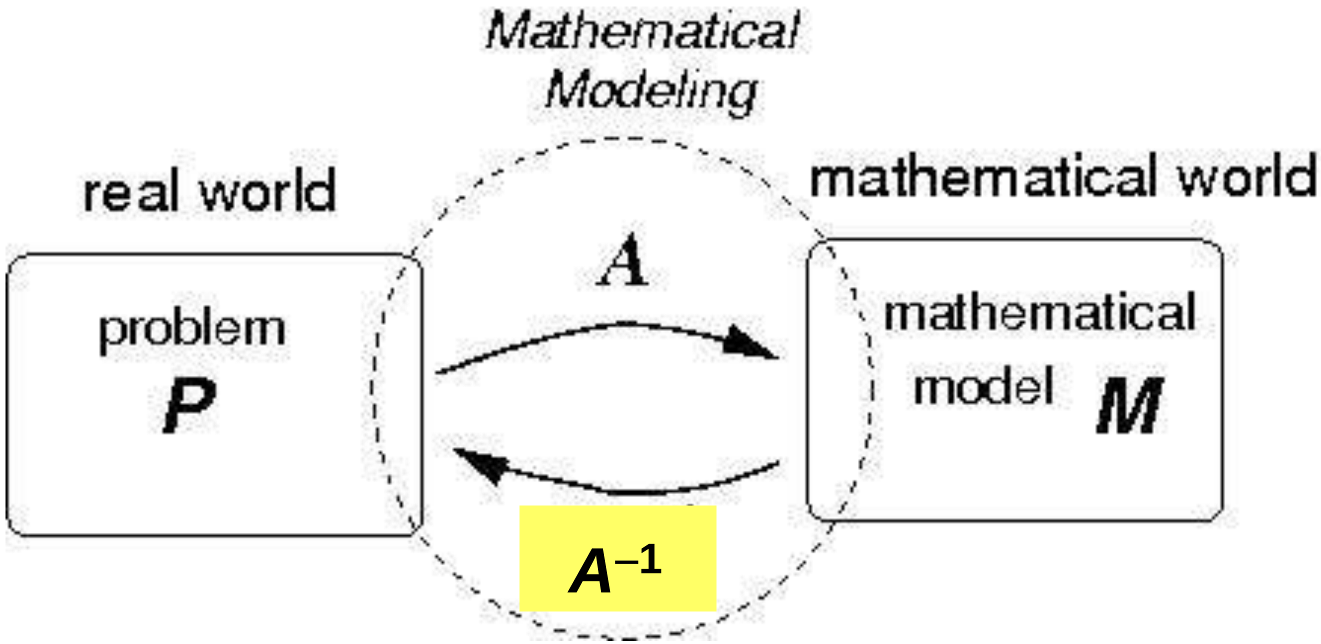
What is the chance on 6?

Model based on

1. **measurements**

2. **mechanical balance laws**

Outcome



Back to real world:

Recommendations

based on

- *qualitative* results
- *quantitative* results

Skills

- **Mathematics**

- **Creativity**
- **Conceptual thinking**
- **Communication skills**

Contents

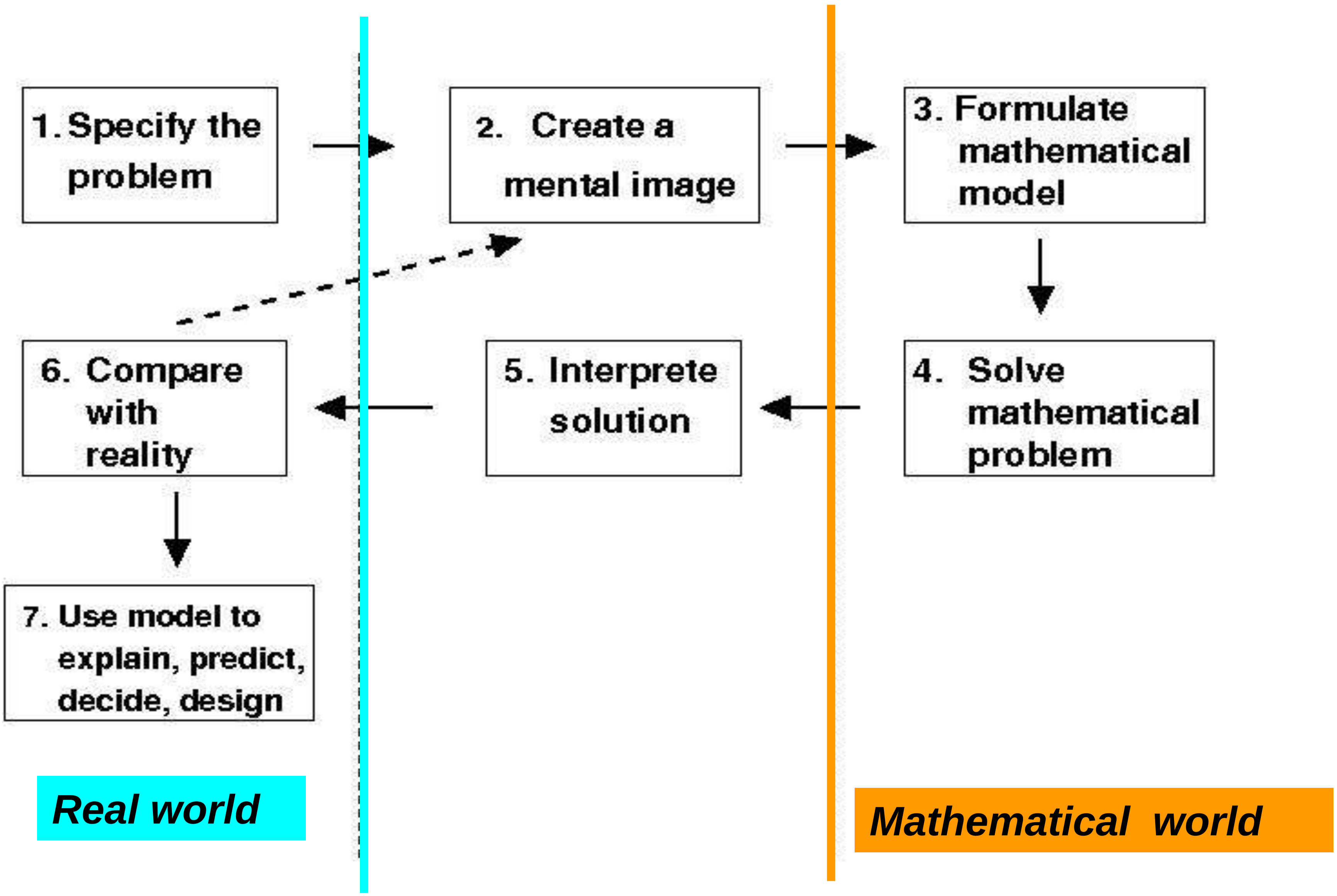
- What is a Mathematical Modeling?
- Aims of Mathematical Modeling
- Mathematical Modeling Cycle

Aims

- Investigate behaviour and relation of elements of problem
- Consider all possibilities, evaluate alternatives, exclude impossibilities
- Verification against measurements; result to be used in other operating regimes

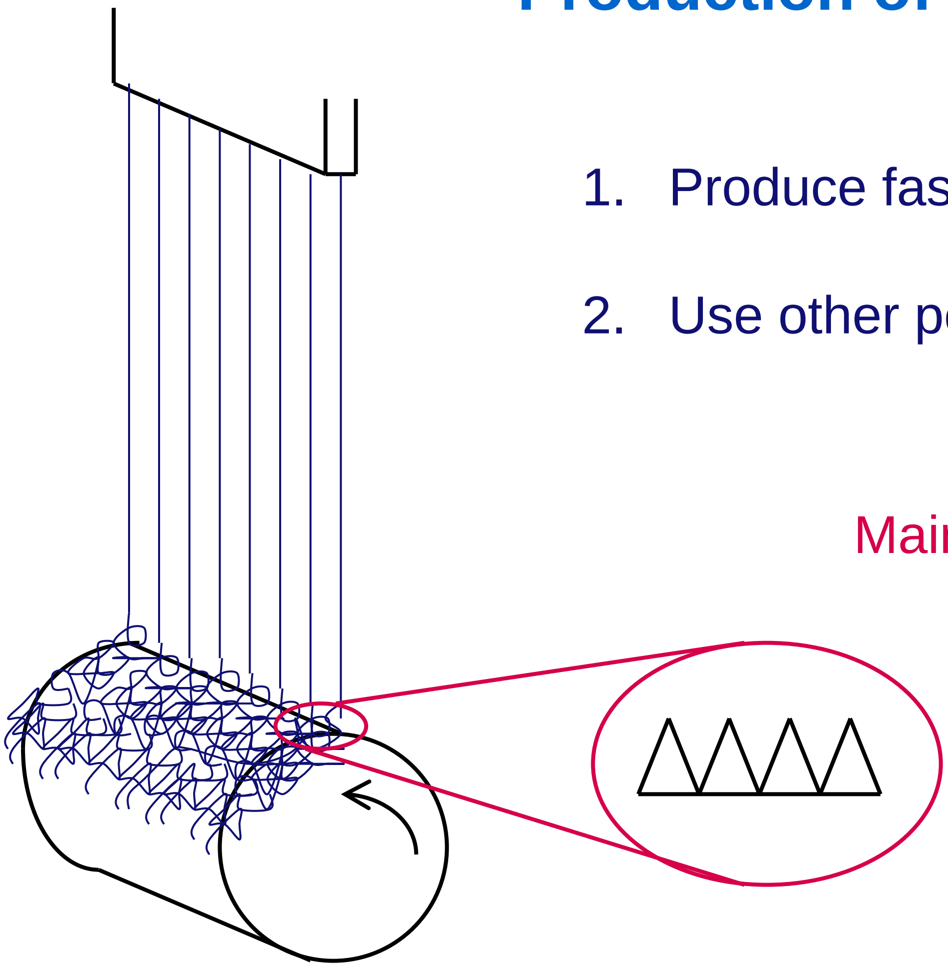
- Optimization
- Facilitate design and proto-typing
- Substantiate decisions

Mathematical Modeling Cycle



Mathematical Modeling Cycle: example

Production of 3D plastic mats



1. Produce faster
2. Use other polymer material (which is cheaper)

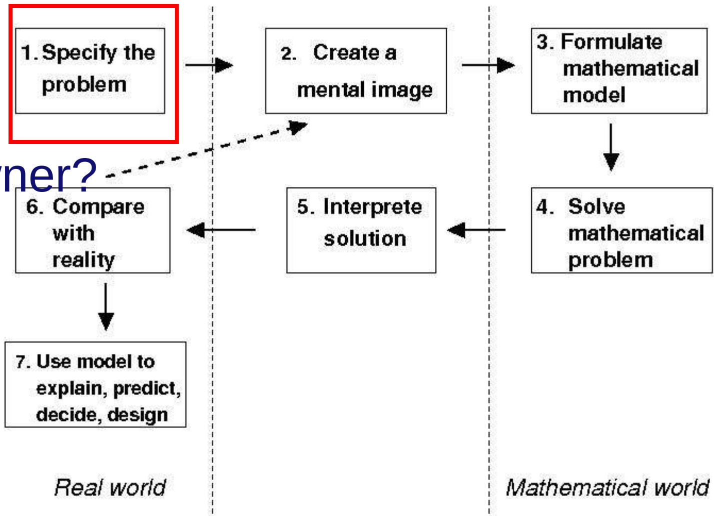
Maintain same mat quality

Step 1: Specify the Problem

- What is background of the problem, its history and its causes.
- What do we want to know?
- Why is one interested in solving the problem? What are the limitations of the current practice?
- What is the solution needed for? What is the purpose of solving the problem?
- What are the constraints to solving the problem?
- Who needs a solution?
- What is the impact/benefit of solving the problem?
- What problem is solved if one has found an answer?
- How will the outcome be judged?
- What form does the solution need to have?

  What do you communicate to the Problem Owner?
- How would you implement the solution?
- Are there other related problems?
- What are the sources of facts and data,

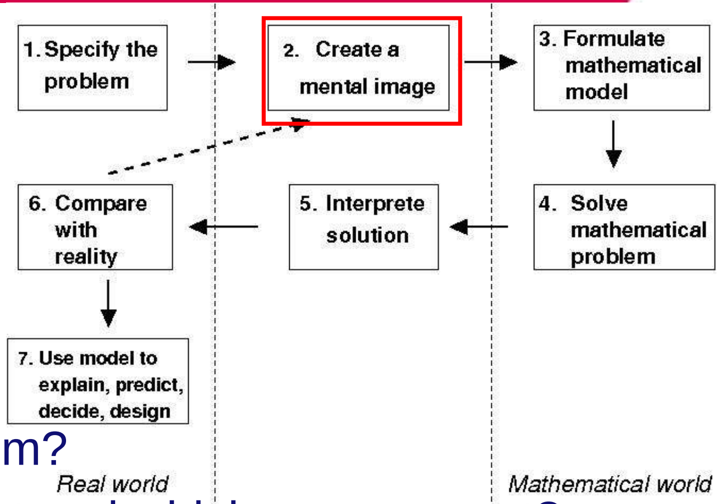
  and are they reliable?



Step 1: Specify the Problem: example

- Rotating drum is also oscillating
- Quality can be characterized by
  - filament diameter in mat
  - amount of attachment of filaments in mat
  - width of coils on drum
- Cheaper polymer material has larger viscosity
- Data of polymer material are available:
  - measurements of shear viscosity and mass density: not accurate
- Data of diameter measured along filament are available
- Form of solution: trends of end-diameter
  - in terms of process/material parameter,
  - based on computer simulation program

Step 2: Create a mental image



- What are the operative processes at work?
- What needs to be determined for solving the problem?
- What are the main features, which ones are relevant and which ones are not?
- What is the relation between the main features?
- Which features do change?
- What is controllable in the problem?
- What are the conditions?
- What are the relevant timescales and dimensions of the problem?
- What kind of material is considered, and what is characteristic of that material?

Simplification → **Assumptions**

Step 2: Create a mental image

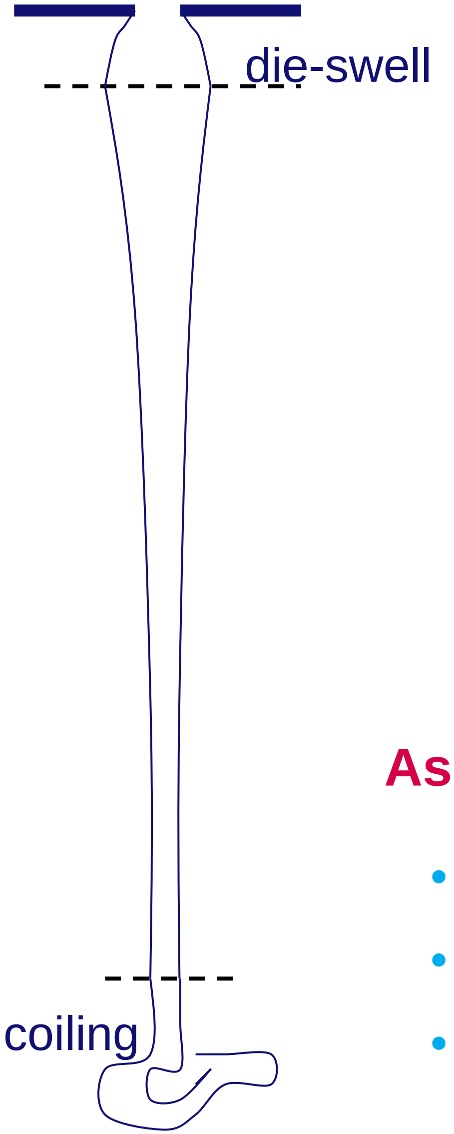
Assumptions ***made in real world***

- **whether or not to include certain features, and why**
- **about the relationships between features,**
- **about their relative effects.**

Decisions: ***In collaboration with Problem Owner***

- **level of detail,**

Step 2: Create a mental image: example



- 1 filament
- polymer is molten, cools down
- begin velocity and diameter at spinneret known
- die-swell, just below spinneret hole
- filament diameter in mat = diameter @ position of coiling
- low velocities: falling under gravity
- thin filament

Assumptions:

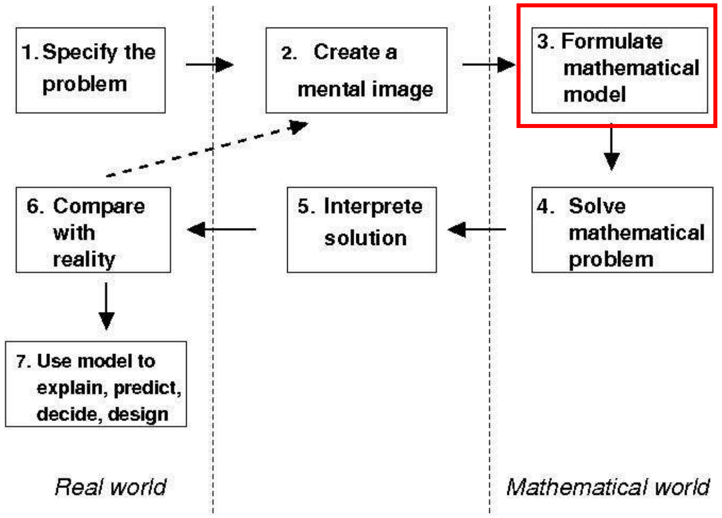
- at coiling position: no influence of surface
- neglect temperature effects
- resistance against stretching

Step 3: Formulate Mathematical Model

Translate relationships of metaphor

into

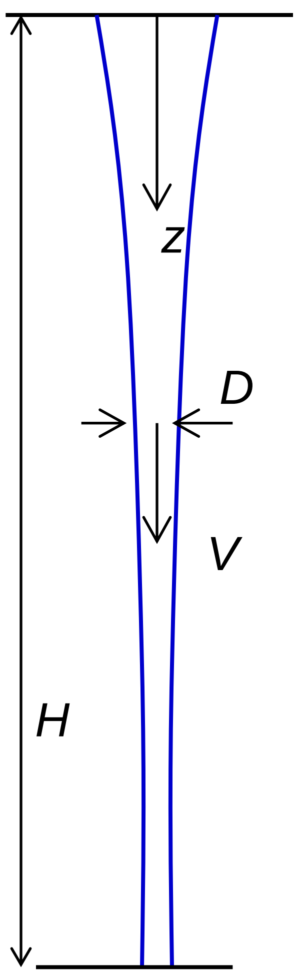
mathematical terms



Mathematical Model:

- **Input and Output**
- **Constants**
- **State variables**
- **Independent variables**
- **Domain**
- **Boundary / initial / constraint conditions**

Step 3: Formulate Mathematical Model: example



Balance of momentum:

Mass balance:

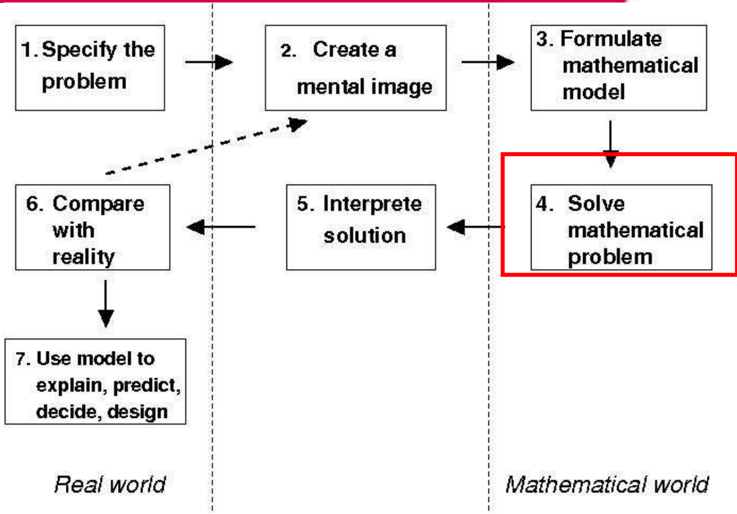
Material behaviour:

State variables: V, D

Boundary conditions: $z = 0, \quad z = H:$

Input parameters:

Step 4: Solve Mathematical Model



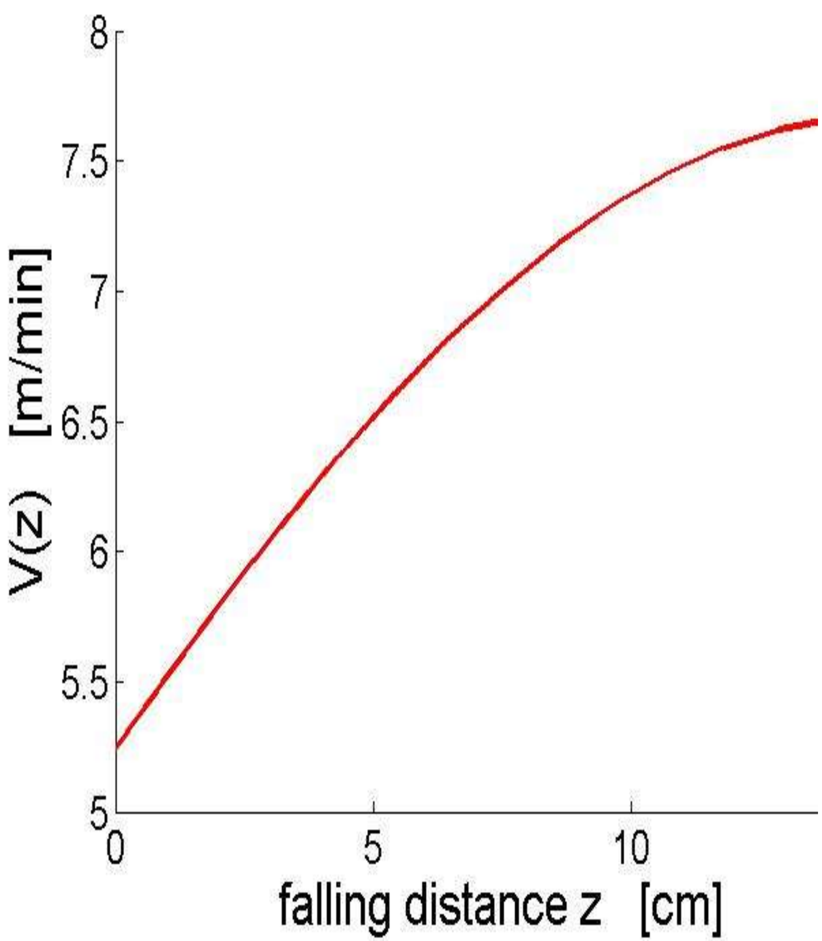
- **Analytically**
- **Numerically**

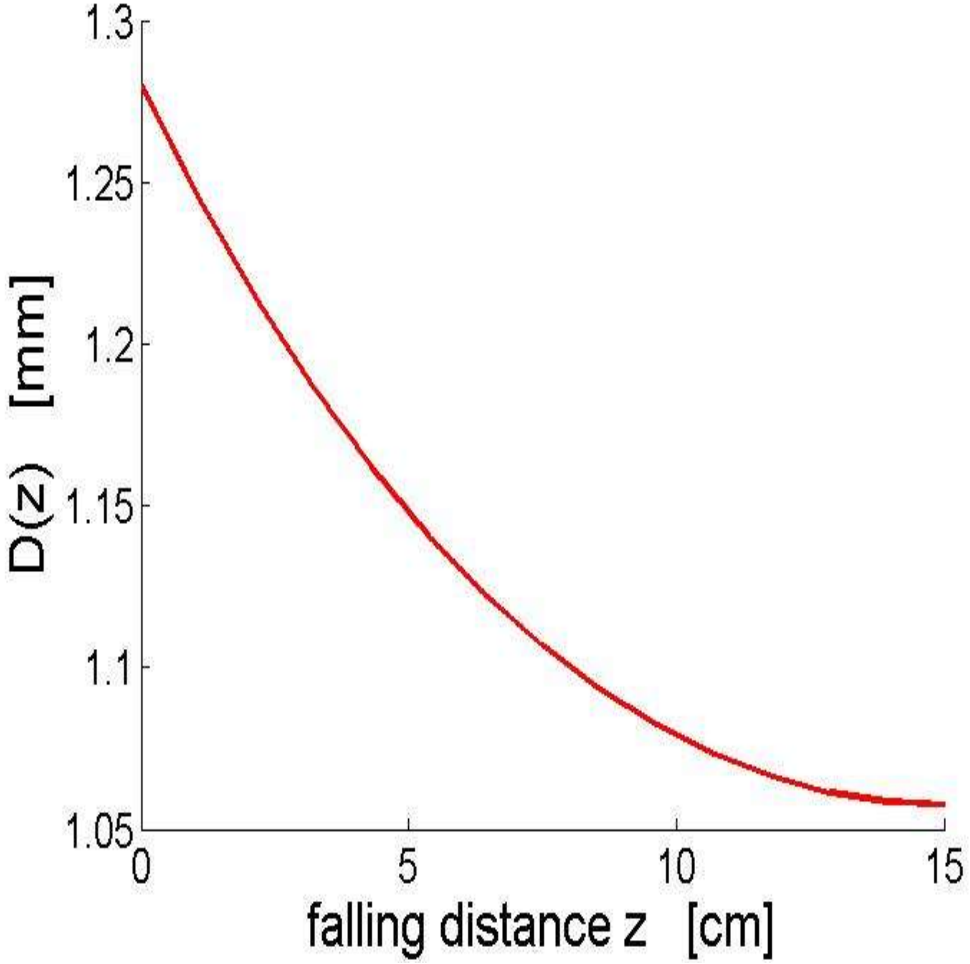
  make equations dimensionless

    such that computations with order-1 numbers

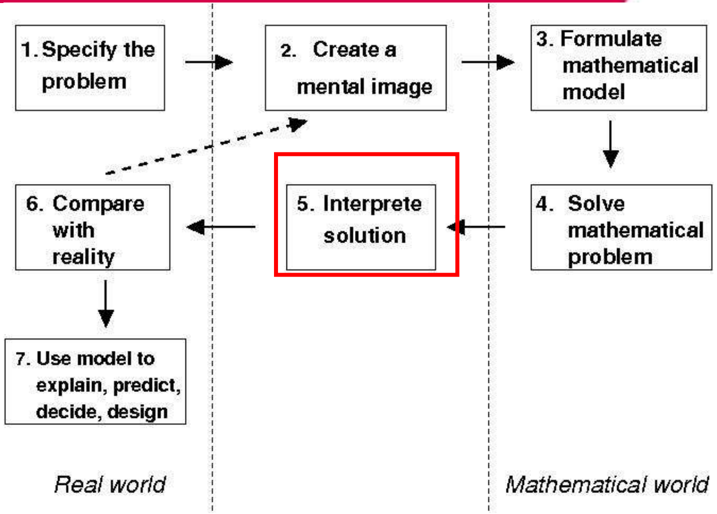
Step 4: Solve Mathematical Model: example

Boundary value problem; numerically, dimensionless





Step 5: Interprete Solution



Retraction of conceptual leap

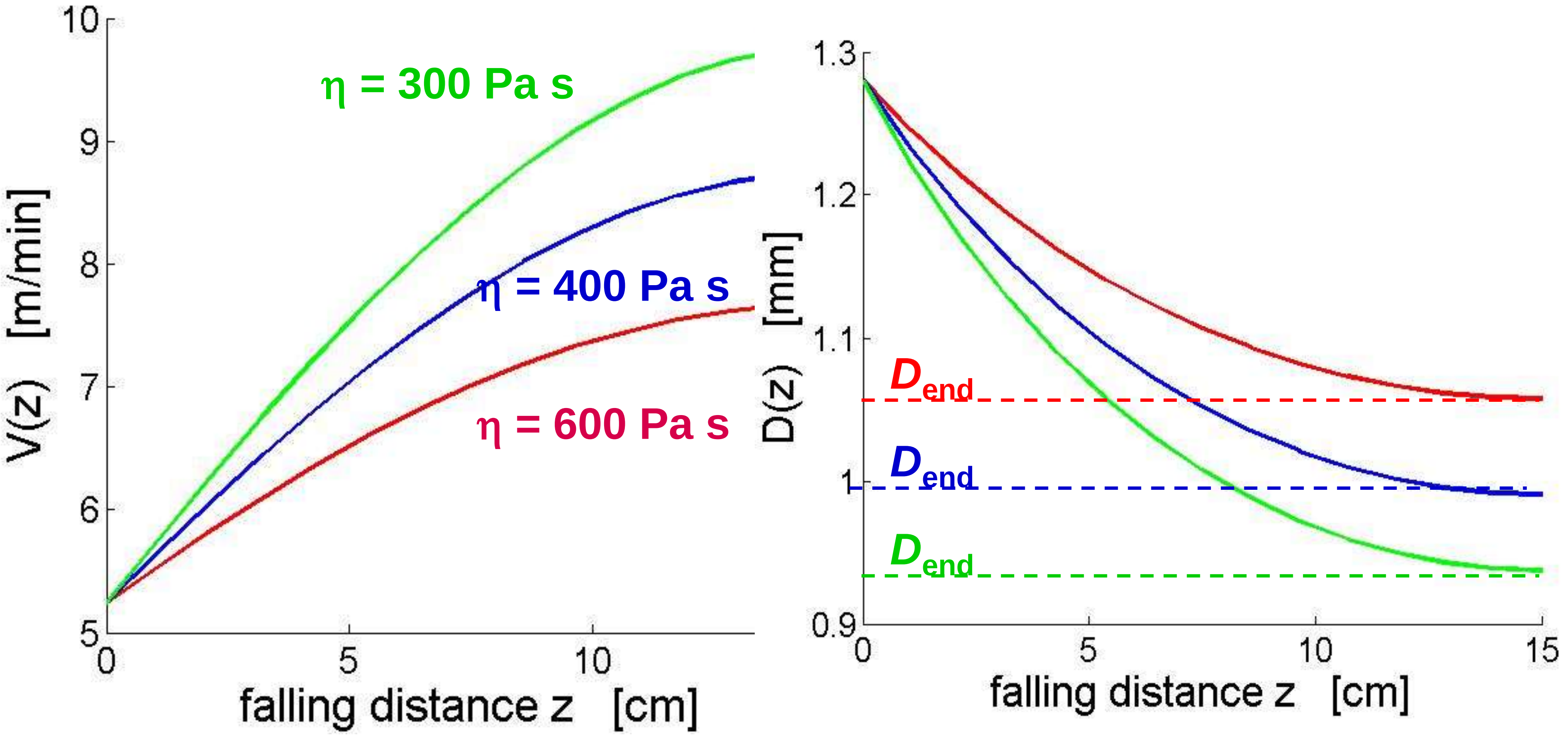
from

mathematical world

to

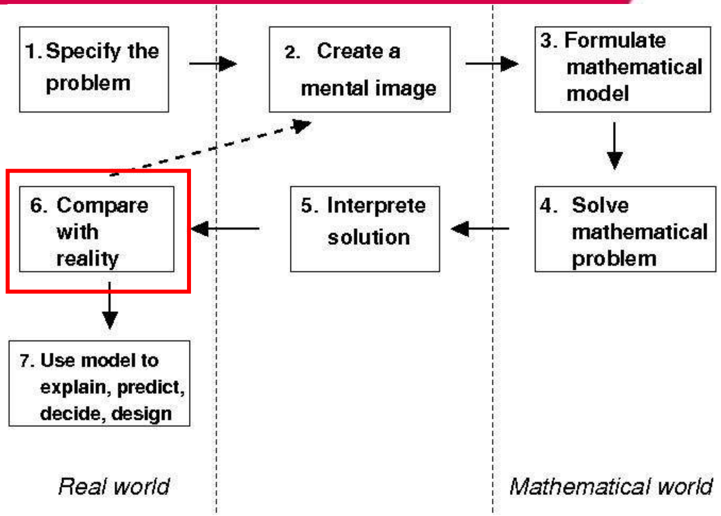
real-world problem

Step 5: Interprete Solution: example



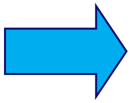
$$D_{\mathrm{end}} = D_{\mathrm{end}}(\Phi, \rho, \eta, H, D_0)$$

Step 6: Compare with Reality



- **Validation of model**
- **Extraction of model parameters**
- **Extension to other operating regimes**

Model insufficient for solving problem

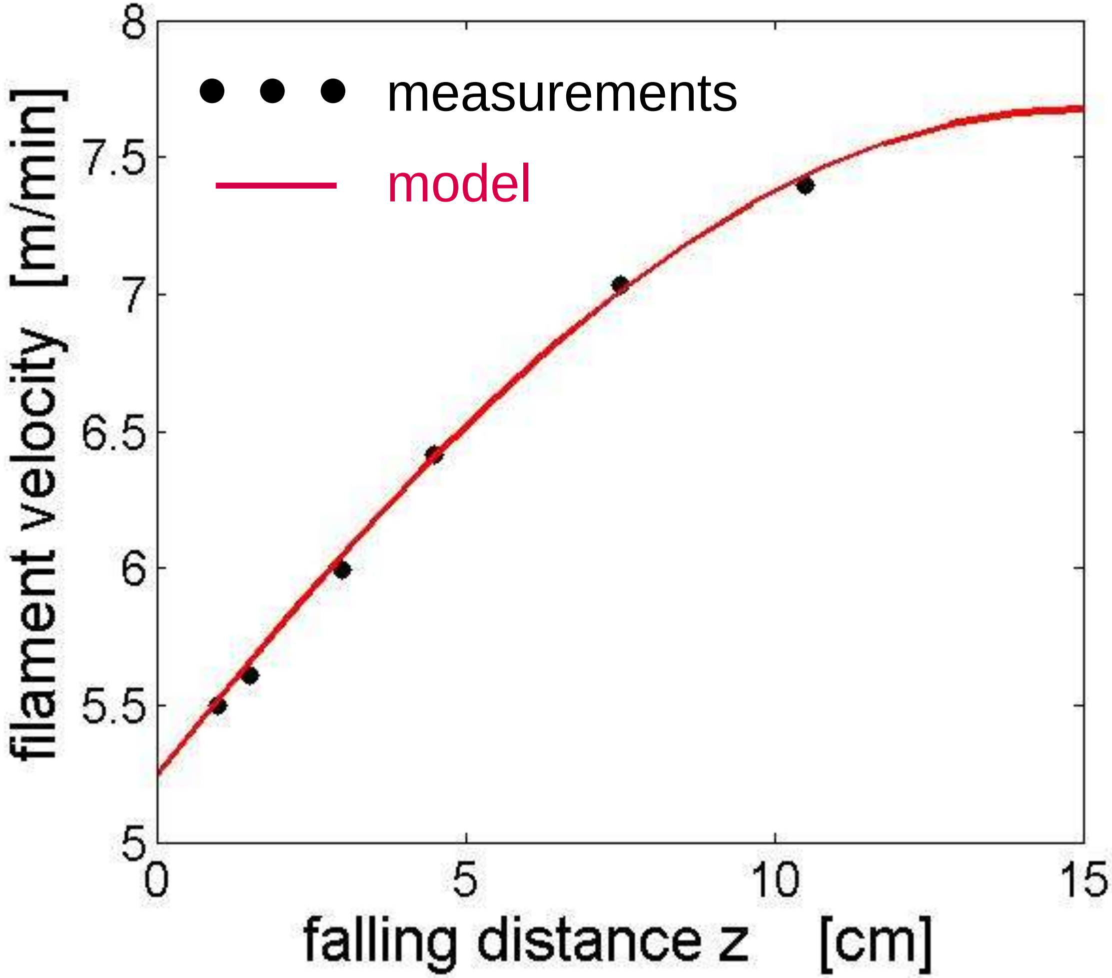


Re-enter modeling cycle again

Accuracy of measurements

Step 6: Compare with Reality

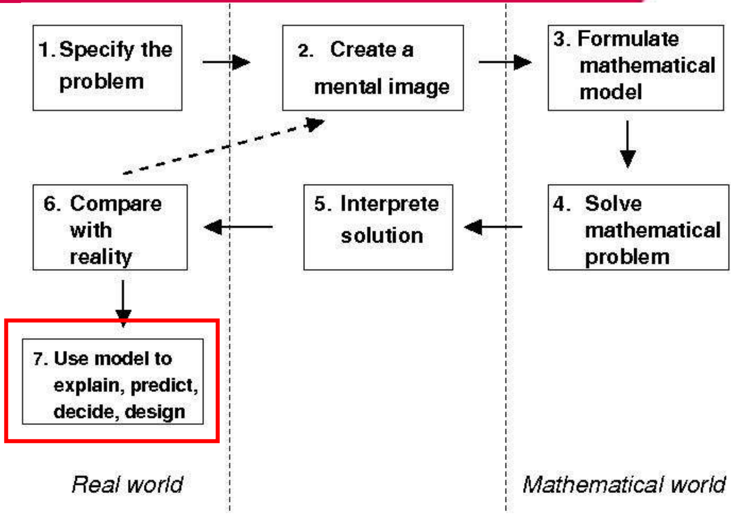
Extraction of material/process parameters:



η = 600 Pa s

D_0 = 1.28 mm

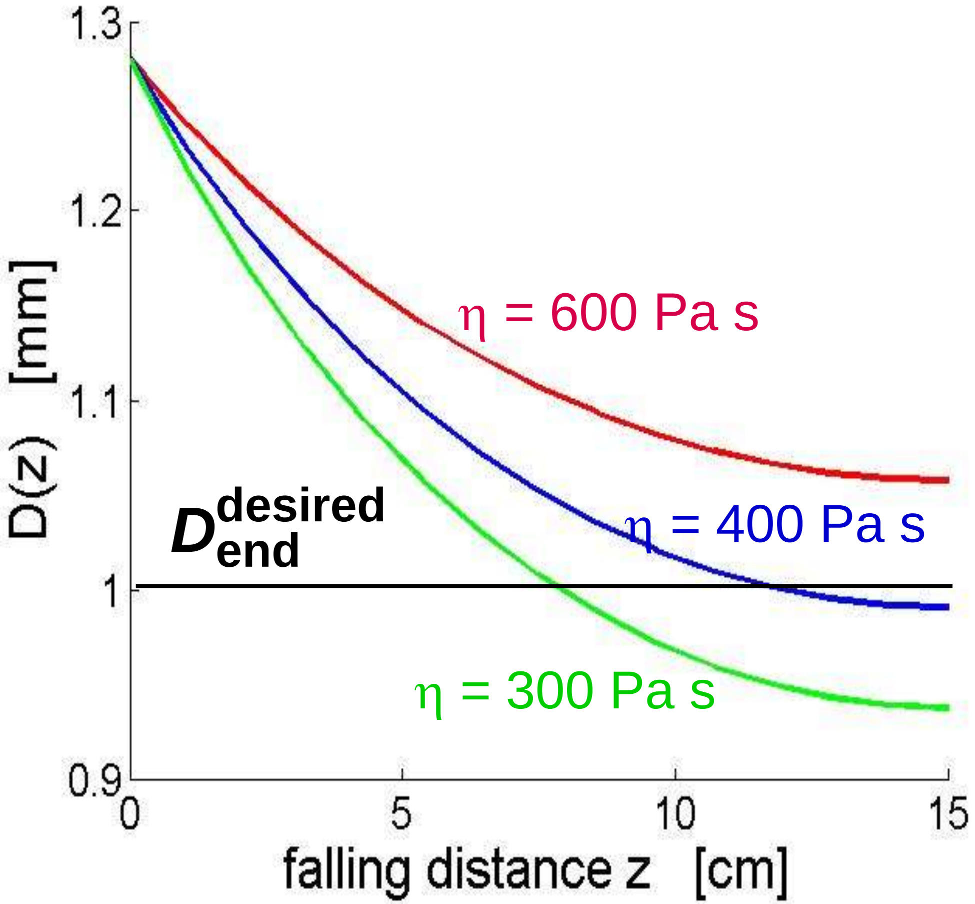
Step 7: Use Model to Explain, Predict, Decide, Desig



- Determine:
  - typical behaviour
  - critical parameters
  - trends
  - dependency on control parameters

Recommendations

Step 7: Use Model to Explain, Predict, Decide, Design



Recommendations

For an end-diameter of 0.4 mm:

- *Increasing the mass flux to 10 gr/min, a falling height is needed of H = 95 cm*
- *For a polymer with η = 800 Pa s, the mass flux must be limited to 5 gr/min, if H must be smaller than 100 cm*

Summary

Mathematical Modeling

- **driven by the problem from the <u>real</u> world;**
- **also includes the <u>interpretation</u> of the solution**

Finding an appropriate model is an art:

Mathematical Modeling:

- ***conceptual thinking***
- ***appropriate simplifications***
- ***close collaboration between Problem Owner and Mathematician.***

Literature

- Saaty TL and Alexander JM, *Thinking with Models,* Pergamon, 1981
- Berry JS, Burghes DN, Huntley ID, James DJG Moscardini AO (Eds.), *Teaching and Applying Mathematical Modelling*, Wiley, 1984
- Houston SK, Blum W, Huntley I, Neill NT (Eds.), *Teaching and Learning Mathematical Modelling,* Albion, 1997

Assignment

For your modeling project :

1. ***Why is a mathematical model needed?***
2. ***Work out Step 1 of the Mathematical Modeling Cycle: "Specify the Problem"***

Hand in with: A.C.T. Aarts (a.c.t.aarts@tue.nl),

Ultimately: Friday 22 October 2010