TECHNISCHE UNIVERSITEIT EINDHOVEN
Department of Mathematics and Computer Science
Exam Stochastic Processes 2 (2S480) on January 20, 2005, 09.00-12.00.

1. Consider a Markov process with states 0, 1 and 2 and with transition rates $q_{0,1} = q_{1,2} = q_{2,0} = \lambda$ and $q_{1,0} = q_{2,1} = q_{0,2} = \mu$.

a. Let T_i $(i = 1, 2)$ denote the time, starting from state i, it takes for the process to enter state 0. Compute $E[T_1]$ and $E[T_2]$.

b. Let S_0 denote the time, starting from state 0, of the first return to state 0. Compute $E[S_0]$.

c. For which values of λ and μ exist the limiting probabilities p_0, p_1 and p_2?

d. Compute the limiting probabilities p_0, p_1 and p_2.

e. Show that

$$p_0 = \frac{1}{\lambda + \mu} \cdot \frac{1}{E[S_0]},$$

and explain this relation.

f. Now consider a Markov process with states $0, 1, \ldots, N$ and with transition rates $q_{0,1} = \cdots = q_{N-1,N} = q_{N,0} = \lambda$ and $q_{1,0} = \cdots = q_{N,N-1} = q_{0,N} = \mu$. What are the limiting probabilities $p_0, \ldots, p_N$? (No calculations required)

2. A machine shop consists of 2 machines. The machines are identical and the lifetimes of the machines are independent and exponentially distributed with parameter λ. At time 0 both machines are functioning. Let $X(t)$ denote the number of machines functioning at time t.

a. Derive the Kolmogorov's backward equations for the transition probabilities $p_{10}(t)$, $p_{21}(t)$ and $p_{20}(t)$.

b. Determine $p_{10}(t)$, $p_{21}(t)$ and $p_{20}(t)$.

c. Let $O_i(t)$ denote the total amount of time that i machines are functioning during the interval $(0, t)$. Compute $E[O_i(t)]$ for $i = 0, 1, 2$.

d. When a machine is functioning, it produces h products per time unit. Compute the mean total number of products produced during the interval $(0, t)$.

3. Consider a branching process $\{X_i, i = 0, 1, 2, \ldots\}$, starting with one particle: $X_0 = 1$. The number of offspring Z of one particle is distributed as

$$P(Z = i) = (1-p)^i p, \qquad i = 0, 1, 2, \ldots$$

a. Give $E[Z]$ and $\text{Var}[Z]$.

b. Compute $P[X_1 = 0]$ and $P[X_2 = 0]$.

c. Determine $E[X_i]$ for $i = 1, 2, \ldots$.

d. For which values of p is extinction of the population certain?

e. Determine the probability that the population dies out (for every $0 \leq p \leq 1$).

4. The lifetime of a machine is exponential with a mean of 1 year. When the machine fails, an emergency unit automatically takes over the production. The unit breaks down after exactly 2 months. Then the whole system (i.e., the machine and the emergency unit) is immediately replaced by a new one. This so-called unplanned replacement costs 1400 euro.

a. At what rate is the system replaced?

b. Determine the long-run average cost per year.

Now, to prevent unplanned replacements, the system is regularly inspected by a repairman. An inspection costs 200 euro. If the machine is still working, nothing is done (the system is still as good as new); otherwise the repairman immediately replaces the whole system, but this preventive replacement is cheaper than an unplanned replacement, namely 600 euro. Assume that the repairman always comes along exactly 10 months after the last event; an event is either an inspection or an unplanned replacement.

c. Determine the probability the repairman has to replace the system.

d. What is the mean time between two successive events?

e. Compute the long-run average cost per year.

Points:

1a	b	c	d	e	f	2a	b	c	d	3a	b	c	d	e	4a	b	c	d	e
3	1	1	2	1	2	2	3	3	2	2	3	1	1	3	1	2	2	2	3