TECHNISCHE UNIVERSITEIT EINDHOVEN
Department of Mathematics and Computer Science
Solutions to Exam Stochastic Processes 2 (2S480) on January 20, 2005, 09.00-12.00.

1.

a.

$$\begin{aligned} E[T_1] &= \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \cdot E[T_2], \\ E[T_2] &= \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} \cdot E[T_1], \end{aligned}$$

yielding

$$E[T_1] = \frac{2\lambda+\mu}{\lambda^2+\lambda\mu+\mu^2}, \qquad E[T_2] = \frac{\lambda+2\mu}{\lambda^2+\lambda\mu+\mu^2}.$$

b.

$$E[S_0] = \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} E[T_1] + \frac{\mu}{\lambda+\mu} E[T_2] = \frac{3}{\lambda+\mu}.$$

c. For all $\lambda, \mu > 0$.

d. The balance equations are:

$$\begin{aligned} p_0(\lambda+\mu) &= p_2\lambda + p_1\mu, \\ p_1(\lambda+\mu) &= p_0\lambda + p_2\mu, \\ p_2(\lambda+\mu) &= p_1\lambda + p_0\mu, \end{aligned}$$

from which, together with $p_0 + p_1 + p_2 = 1$, follows that

$$p_1 = p_2 = p_3 = \frac{1}{3}.$$

e.

$$p_0 = \frac{1}{3} = \frac{1}{\lambda+\mu} \cdot \frac{1}{\frac{3}{\lambda+\mu}} = \frac{1}{\lambda+\mu} \cdot \frac{1}{E[S_0]};$$

clearly, the fraction of time spent in state 0 is equal to the expected time spent in 0 during a cycle divided by the expected cycle length.

f. By symmetry, $p_0 = \ldots = p_N = \frac{1}{N+1}$.

2.

a.

$$\begin{aligned} p'_{10}(t) &= \lambda - \lambda p_{10}(t), \\ p'_{21}(t) &= 2\lambda p_{11}(t) - 2\lambda p_{21}(t), \\ p'_{20}(t) &= 2\lambda p_{10}(t) - 2\lambda p_{20}(t). \end{aligned}$$

b.

$$\begin{aligned} p_{10}(t) &= 1 - p_{11}(t) \ = \ 1 - e^{-\lambda t}, \\ p_{21}(t) &= 2e^{-\lambda t} - 2e^{-2\lambda t} \ = \ 2e^{-\lambda t}(1 - e^{-\lambda t}), \\ p_{20}(t) &= 1 - 2e^{-\lambda t} + e^{-2\lambda t} \ = \ (1 - e^{-\lambda t})^2. \end{aligned}$$

c.

$$E(O_i(t)) = \int_{s=0}^{t} p_{2i}(s)\mathrm{d}s, \qquad i = 0, 1, 2,$$

so we find

$$\begin{aligned} E(O_2(t)) &= \frac{1}{2\lambda}(1 - e^{-2\lambda t}), \\ E(O_1(t)) &= \frac{2}{\lambda}(1 - e^{-\lambda t}) + \frac{1}{\lambda}(1 - e^{-2\lambda t}), \\ E(O_0(t)) &= t - E(O_1(t)) - E(O_2(t)) \ = \ t - \frac{2}{\lambda}(1 - e^{-\lambda t}) + \frac{1}{2\lambda}(1 - e^{-2\lambda t}). \end{aligned}$$

d. The mean total production in $(0, t)$ is $2hE(O_2(t)) + hE(O_1(t))$.

3.

a. $E[Z] = \frac{1-p}{p}$ and $\mathrm{Var}[Z] = \frac{1-p}{p^2}$.

b. $P[X_1 = 0] = p$ and

$$P[X_2 = 0] = \sum_{i=0}^{\infty} P[Z = i](P[X_1 = 0])^i = \sum_{i=0}^{\infty} p((1-p)p)^i = \frac{p}{1 - (1-p)p}.$$

c. $E[X_i] = E[Z]^i = \frac{(1-p)^i}{p^i}$.

d. For all p for which $E[Z] \leq 1$, and thus for all $\frac{1}{2} \leq p \leq 1$.

e. Clearly $\pi_0 = 1$ for $\frac{1}{2} \le p \le 1$. If $0 \le p < \frac{1}{2}$, then π_0 is the root in $[0,1)$ of

$$\pi_0 = \sum_{i=0}^{\infty} P[Z = i]\pi_0^i = \frac{p}{1-(1-p)\pi_0},$$

yielding

$$\pi_0 = \frac{p}{1-p}.$$

4.

a. The mean time between replacements is $\frac{7}{6}$ year. Hence the system is replaced $\frac{6}{7}$ times per year.

b. The long-run average cost per year is $1400 \cdot \frac{6}{7} = 1200$ euro.

c. Let L denote the lifetime of the machine, so L is exponential with mean 1. The probability that an event is an inspection is $P[L > \frac{2}{3}] = e^{-2/3} = 0.513$ and the probability that an event is an inspection *and* the machine is still working when it is inspected is $P[L > \frac{5}{6}] = e^{-5/6} = 0.435$. Hence, the probability that the repairman has to replace the system is

$$1 - \frac{P[L > \frac{5}{6}]}{P[L > \frac{2}{3}]} = 0.154.$$

d. The mean time between two events is

$$\int_{t=0}^{\frac{2}{3}} (t + \frac{1}{6})e^{-t}\mathrm{d}t + \frac{5}{6}\int_{t=\frac{2}{3}}^{\infty} e^{-t}\mathrm{d}t = \frac{7}{6} - e^{-2/3} = 0.653.$$

e. The mean cost of an event is $1400 \cdot P[L < \frac{2}{3}] + 200 \cdot P[L > \frac{2}{3}] + 600 \cdot P[\frac{2}{3} < L < \frac{5}{6}] = 831$ euro. Hence the mean cost per year is

$$\frac{831}{0.653} = 1273 \text{ euro.}$$

Points:

1a	b	c	d	e	f	2a	b	c	d	3a	b	c	d	e	4a	b	c	d	e
3	1	1	2	1	2	2	3	3	2	2	3	1	1	3	1	2	2	2	3