TECHNISCHE UNIVERSITEIT EINDHOVEN
Department of Mathematics and Computer Science
Exam Stochastic Processes 2 (2S480) on November 20, 2006, 14.00-17.00.

**1.** A store keeps 2 items on stock. Customers arrive at a Poisson rate $\lambda$. Each customer demands 1 item, and each time an item is sold, the store immediately orders a new one. The delivery lead time is exponentially distributed with rate $\mu$. Any demand that cannot be immediately satisfied is lost. Let $X(t)$ denote the number of items on stock at time $t$.

a. Argue that $\{X(t), t \geq 0\}$ is a continuous time Markov chain and give its transition rates.

b. Suppose that at time 0 there are exactly 2 items on stock. Compute the expected time until the stock is empty for the first time.

c. What proportion of time the stock is empty?

d. What proportion of customer demand is lost?

Suppose now that the store only orders new items when the stock is empty and then orders 2 items, the delivery time of which is exponentially distributed with rate $\mu$.

e. What proportion of time the stock is empty?

Now suppose that at time 0 there are exactly 2 items on stock and that the store decides to stop ordering new items.

f. Derive the Kolmogorov's backward equations for the transition probabilities $P_{20}(t)$ and $P_{10}(t)$ of the continuous time Markov chain $\{X(t), t \geq 0\}$.

g. Determine $P_{20}(t)$.

**2.** Customers arrive at a single-service facility at a Poisson rate $\lambda$. The service time for each customer is exponentially distributed with rate $\mu$. At time 0 there is exactly 1 customer in the system. Let $Z$ denote the number of arrivals during a service time.

a. Show that
$$G(r) = \sum_{j=0}^{\infty} P(Z = j) r^j = \frac{\mu}{\mu + \lambda(1-r)}, \qquad |r| \leq 1.$$

b. Compute $E[Z]$, $E[Z(Z-1)]$ and $\text{Var}[Z]$.

c. Compute the probability that the system ever becomes empty again for all possible values of $\lambda$ and $\mu$. (Hint: customers arriving during the service of the first customer are generation 1; customers arriving during the service of generation 1 customers are generation 2 and so on.)

**3.** Orders arrive at a manufacturing system at a Poisson rate of $\lambda$ per hour, where they are produced in batches. After finishing a batch, all waiting orders are produced in the next batch as soon as $N$ or more orders are present. While waiting for sufficiently many orders, the manufacturing system is idle, which costs $w$ euro per hour. The set-up costs of a batch are $K$ euro. The batch production time (in hours) is uniformly distributed on $(0, h)$, independent of the number of orders in the batch.

a. Let $p_k$ denote the probability that $k$ orders are waiting immediately after finishing a batch. Show that

$$p_k = \frac{1}{\lambda h}\left[1 - e^{-\lambda h}\sum_{i=0}^{k}\frac{(\lambda h)^i}{i!}\right].$$

b. Determine the long-run average number of batches produced per hour.

c. Determine the long-run average costs per hour.

**4.** Two classes of customers arrive at a single-server facility at a Poisson rate $\lambda_1$ and $\lambda_2$, respectively. Class 1 customers are always accepted upon arrival, but class 2 customers are only accepted if the number of customers in the system is less than $N$. If the system contains $N$ or more customers, arriving class 2 customers are rejected and sent to another service facility. The service times are exponentially distributed with rate $\mu$.

a. What conditions on $\lambda_1$, $\lambda_2$ and $\mu$ are necessary to guarantee that the system is stable?

b. Define states and set up the balance equations for the limiting probabilities (do not solve yet).

c. In terms of the probabilities in part b, what is the fraction of class 2 customers that is accepted?

d. In terms of the probabilities in part b, what is the average number in the system and the average amount of time an *accepted* customer spends in the system?

Suppose $\lambda_1 = 1$, $\lambda_2 = 3$, $\mu = 2$ and $N = 4$.

e. Solve the balance equations for the limiting probabilities.

**Points:**

| 1a | b | c | d | e | f | g | 2a | b | c | 3a | b | c | 4a | b | c | d | e |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 3 | 3 | 2 | 2 | 2 | 2 | 2 |