TECHNISCHE UNIVERSITEIT EINDHOVEN
Department of Mathematics and Computer Science
Solutions to Exam Stochastic Processes 2 (2S480) on November 20, 2006, 14.00-17.00.

1.

a. The transition rate matrix Q is

$$\begin{pmatrix} -2\mu & 2\mu & 0 \\ \lambda & -\lambda-\mu & \mu \\ 0 & \lambda & -\lambda \end{pmatrix}.$$

b. Let T_i ($i = 1, 2$) denote the time, starting from state i, it takes to enter state 0. Then

$$E[T_1] = \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} E[T_2], \qquad E[T_2] = \frac{1}{\lambda} + E[T_1].$$

Solving these equations gives

$$E[T_2] = \frac{2}{\lambda} + \frac{\mu}{\lambda^2}.$$

c. The balance equations are:

$$\begin{aligned} 2\mu P_0 &= \lambda P_1, \\ \mu P_1 &= \lambda P_2, \end{aligned}$$

from which, together with the normalization equation, follows that

$$P_0 = \frac{\lambda^2}{\lambda^2 + 2\lambda\mu + 2\mu^2}.$$

d. P_0.

e. Now the transition rate matrix Q is

$$\begin{pmatrix} -\mu & 0 & \mu \\ \lambda & -\lambda & 0 \\ 0 & \lambda & -\lambda \end{pmatrix}.$$

So the balance equations are:

$$\begin{aligned} \mu P_0 &= \lambda P_1, \\ \lambda P_1 &= \lambda P_2, \end{aligned}$$

from which, together with the normalization equation, follows that

$$P_0 = \frac{\lambda}{\lambda + 2\mu}.$$

f.

$$\begin{aligned} P'_{20}(t) &= \lambda P_{10}(t) - \lambda P_{20}(t), \\ P'_{10}(t) &= \lambda - \lambda P_{10}(t). \end{aligned}$$

g.

$$P_{20}(t) = 1 - e^{-\lambda t} - \lambda t e^{-\lambda t}.$$

2.

a. Since

$$P(Z = j) = \left(\frac{\lambda}{\lambda+\mu}\right)^j \frac{\mu}{\lambda+\mu},$$

we get

$$G(r) = \sum_{j=0}^{\infty} \left(\frac{\lambda r}{\lambda+\mu}\right)^j \frac{\mu}{\lambda+\mu} = \frac{\mu}{\lambda+\mu} \cdot \frac{1}{1-\lambda r/(\lambda+\mu)} = \frac{\mu}{\mu+\lambda(1-r)}.$$

b. We have

$$E[Z] = G'(1) = \frac{\lambda}{\mu}, \qquad E[Z(Z-1)] = G''(1) = 2\frac{\lambda^2}{\mu^2},$$

and thus,

$$\text{Var}[Z] = E[Z(Z-1)] + E[Z] - E[Z]^2 = \frac{\lambda^2}{\mu^2} + \frac{\lambda}{\mu}.$$

c. Let π_0 denote the probability that the system ever becomes empty again. Then $\pi_0 = 1$ for all $\lambda \leq \mu$. If $\lambda > \mu$, then π_0 is the unique root on $(0, 1)$ of the equation

$$\pi_0 = G(\pi_0),$$

yielding

$$\pi_0 = \frac{\mu}{\lambda}.$$

3.

a. By conditioning on the duration of the batch service time, we get

$$p_k = \int_0^h e^{-\lambda t} \frac{(\lambda t)^k}{k!} \frac{dt}{h} = \frac{1}{\lambda h} \int_0^h \lambda e^{-\lambda t} \frac{(\lambda t)^k}{k!} dt = \frac{1}{\lambda h} \left[1 - e^{-\lambda h} \sum_{i=0}^{k} \frac{(\lambda h)^i}{i!}\right].$$

b. A cycle is the time between the production start of two successive batches, so

$$E[\text{length of a cycle}] = \frac{h}{2} + \sum_{k=0}^{N-1} p_k \frac{N-k}{\lambda},$$

and thus, the long-run average number of batches produced per hour is

$$\frac{1}{E[\text{length of a cycle}]} = \frac{1}{h/2 + \sum_{k=0}^{N-1} p_k (N-k)/\lambda}.$$

c. The long-run average costs per hour are equal to

$$\frac{E[\text{cost incurred during a cycle}]}{E[\text{length of a cycle}]} = \frac{K + w\sum_{k=0}^{N-1} p_k (N-k)/\lambda}{h/2 + \sum_{k=0}^{N-1} p_k (N-k)/\lambda}.$$

4.

a. $\lambda_1 < \mu$.

b. The states are n, $n \geq 0$, refering to the total number in the system. The balance equations are

$$\begin{aligned} (\lambda_1 + \lambda_2)P_n &= \mu P_{n+1}, \qquad n = 0, 1, \ldots, N-1, \\ \lambda_1 P_n &= \mu P_{n+1}, \qquad n \geq N. \end{aligned}$$

c. By the PASTA property we have

$$P_{acc} = \sum_{n=0}^{N-1} p_n.$$

d. The average number in the system is

$$L = \sum_{n=0}^{\infty} n P_n$$

and by Little's law,

$$W = \frac{L}{\lambda},$$

where λ is the number of accepted customers per time unit, so

$$\lambda = \lambda_1 + \lambda_2 P_{acc}.$$

e.

$$
\begin{aligned}
P_n &= 2^n P_0, \qquad n = 0, \ldots, 4, \\
P_n &= \left(\frac{1}{2}\right)^{n-4} 2^4 P_0, \qquad n > 4, \\
P_0 &= \frac{1}{47}.
\end{aligned}
$$

Points:

1a	b	c	d	e	f	g	2a	b	c	3a	b	c	4a	b	c	d	e
1	2	1	1	1	2	2	3	3	4	4	3	3	2	2	2	2	2