TECHNISCHE UNIVERSITEIT EINDHOVEN
Department of Mathematics and Computer Science
Exam Stochastic Processes 2 (2S480) on November 24, 2004, 09.00-12.00.

1. Consider a Markov process with states 0, 1 and 2 and with the following transition rate matrix Q:

$$\begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -\lambda-\mu & \lambda \\ \mu & 0 & -\mu \end{pmatrix}$$

where $\lambda > 0$ and $\mu > 0$.

a. Let T_i $(i = 1, 2)$ denote the time, starting from state i, it takes for the process to enter state 0. Compute $E[T_1]$ and $E[T_2]$.

b. Let S_0 denote the time, starting from state 0, of the first return to state 0. Compute $E[S_0]$.

c. For which values of λ and μ exist the limiting probabilities p_0, p_1 and p_2?

d. Compute the limiting probabilities p_0, p_1 and p_2.

e. Show that

$$p_0 = \frac{1}{\lambda} \cdot \frac{1}{E[S_0]},$$

and explain this relation.

2. Consider an on-off source: this is source that is alternatingly on and off. The on times are exponential with parameter α and the off times are exponential with parameter β. Let $X(t)$ denote the state of the source at time t (with state 1 meaning that the source is on and with state 0 that it is off).

a. Give the transition rate matrix Q of the process $\{X(t), t \geq 0\}$.

b. Derive the Kolmogorov's forward equations for the transition probabilities $p_{11}(t)$ and $p_{10}(t)$.

c. Show that $p_{11}(t)$ satisfies the differential equation

$$p'_{11}(t) = \beta - p_{11}(t)(\alpha + \beta), \qquad t \geq 0,$$

and determine $p_{11}(t)$ and $p_{10}(t)$.

d. Let $O(t)$ denote the total amount of time the source is on during the interval $(0, t)$. Compute $E[O(t)]$ when the source is on at time $t = 0$.

e. Determine the long-run fraction of time the source is on.

3. Consider a branching process $\{X_i, i = 0, 1, 2, \ldots\}$, starting with one particle: $X_0 = 1$. The number of offspring Z of one particle is binomial with parameters n and p (where n is the number of trials and p is the probability of success).

a. Give $E[Z]$ and $\text{Var}[Z]$.

b. Compute $P[X_1 = 0]$ and $P[X_2 = 0]$.

c. Determine $E[X_i]$ for $i = 1, 2, \ldots$.

d. For which values of n and p is extinction of the population certain?

e. Let $n = 2$. Determine the probability that the population dies out (for every $0 \leq p \leq 1$).

4. Consider again the on-off source in **2**. While the source is on arrivals are generated according to a Poisson process with rate λ; there are no arrivals when the source is off (this is called an *interrupted Poisson process*). Let $m_1(t)$ and $m_0(t)$ denote the mean number of arrivals in $(0, t)$ when the source is on, respectively off at time $t = 0$; the Laplace-Stieltjes transform of $m_i(t)$ is defined as

$$\mu_i(s) = \int_{t=0}^{\infty} e^{-st} \mathrm{d}m_i(t).$$

a. Show that $m_1(t)$ and $m_0(t)$ satisfy the equations (with $\gamma = \alpha + \lambda$)

$$\begin{aligned} m_1(t) &= \frac{\lambda}{\gamma}\left(1 - e^{-\gamma t}\right) + \frac{\lambda}{\gamma}\int_{x=0}^{t} m_1(t-x)\gamma e^{-\gamma x}\mathrm{d}x + \frac{\alpha}{\gamma}\int_{x=0}^{t} m_0(t-x)\gamma e^{-\gamma x}\mathrm{d}x, \\ m_0(t) &= \int_{x=0}^{t} m_1(t-x)\beta e^{-\beta x}\mathrm{d}x. \end{aligned}$$

(Hint: condition on the first *event*, i.e., an arrival or the source changes state).

b. Show that

$$\mu_1(s) = \frac{\lambda\beta}{(\alpha+\beta)s} + \frac{\lambda\alpha}{(\alpha+\beta)(\alpha+\beta+s)}.$$

c. Determine $m_1(t)$.

d. What is the relation between $E[O(t)]$ in **2**d and $m_1(t)$?

Points:

1a	b	c	d	e	2a	b	c	d	e	3a	b	c	d	e	4a	b	c	d
3	1	1	3	2	1	2	3	2	2	1	3	2	1	3	3	3	2	2