TECHNISCHE UNIVERSITEIT EINDHOVEN
Department of Mathematics and Computer Science
Solutions to Exam Stochastic Processes 2 (2S480) on November 24, 2004, 09.00-12.00.

**1.**

a. $E[T_2] = 1/\mu$ and

$$E[T_1] = \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \cdot E[T_2] = \frac{1}{\mu}.$$

b.

$$E[S_0] = \frac{1}{\lambda} + E[T_1] = \frac{1}{\lambda} + \frac{1}{\mu}.$$

c. For all $\lambda, \mu > 0$.

d. The balance equations are:

$$\begin{aligned} p_0\lambda &= p_1\mu + p_2\mu, \\ p_1(\lambda+\mu) &= p_0\lambda, \\ p_2\mu &= p_1\lambda, \end{aligned}$$

from which follows that

$$p_1 = \frac{\lambda}{\lambda+\mu} \cdot p_0, \qquad p_2 = \frac{\lambda}{\mu} \cdot \frac{\lambda}{\lambda+\mu} \cdot p_0,$$

and

$$p_0 = \left(1 + \frac{\lambda}{\lambda+\mu} + \frac{\lambda}{\mu} \cdot \frac{\lambda}{\lambda+\mu}\right)^{-1} = \left(1 + \frac{\lambda}{\mu}\right)^{-1}.$$

e.

$$p_0 = \frac{1}{1+\frac{\lambda}{\mu}} = \frac{1}{\lambda} \cdot \frac{1}{\frac{1}{\lambda}+\frac{1}{\mu}} = \frac{1}{\lambda} \cdot \frac{1}{E[S_0]};$$

clearly, the fraction of time spent in state 0 is equal to the expected time spent in 0 during a cycle divided by the expected cycle length.

**2.**

a.

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}.$$

b.

$$
\begin{aligned}
p'_{11}(t) &= p_{10}(t)\beta - p_{11}(t)\alpha, \\
p'_{10}(t) &= p_{11}(t)\alpha - p_{10}(t)\beta.
\end{aligned}
$$

c. Substitute $p_{10}(t) = 1 - p_{11}(t)$ in the differential equation for $p_{11}(t)$. Together with the initial condition $p_{11}(0) = 1$, this differential equation is solved by

$$p_{11}(t) = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta} e^{-(\alpha+\beta)t},$$

and thus

$$p_{10}(t) = 1 - p_{11}(t) = \frac{\alpha}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta} e^{-(\alpha+\beta)t}.$$

d.

$$E[O(t)] = \int_{s=0}^{t} p_{11}(s)\mathrm{d}s = \frac{\beta}{\alpha+\beta} \cdot t + \frac{\alpha}{(\alpha+\beta)^2}\left(1 - e^{-(\alpha+\beta)t}\right).$$

e. $P_{on}$ is equal to $\lim_{t\to\infty} p_{11}(t)$, or it follows from

$$P_{on} = \frac{E[\text{on time}]}{E[\text{on time}] + E[\text{off time}]} = \frac{\frac{1}{\alpha}}{\frac{1}{\alpha} + \frac{1}{\beta}} = \frac{\beta}{\alpha+\beta}.$$

**3.**

a. $E[Z] = np$ and $\mathrm{Var}[Z] = np(1-p)$.

b. $P[X_1 = 0] = (1-p)^n$ and

$$P[X_2 = 0] = \sum_{k=0}^{n} P[Z = k](P[X_1 = 0])^k = (1 - p + p(1-p)^n)^n.$$

c. $E[X_i] = (np)^i$.

d. For all $np \le 1$.

e. Let $n = 2$. Then $\pi_0 = 1$ for $0 \le p \le \frac{1}{2}$. If $\frac{1}{2} < p \le 1$, then $\pi_0$ is the root in $[0,1)$ of

$$\pi_0 = (1-p)^2 + 2p(1-p)\pi_0 + p^2\pi_0^2,$$

yielding

$$\pi_0 = \left(\frac{1-p}{p}\right)^2.$$

**4.**

a. Let $X$ denote the time till the first event, when the source is on at time $t = 0$. Then $X$ is exponential with parameter $\gamma = \alpha + \lambda$, and the event is an arrival with probability $\lambda/\gamma$ and otherwise the source is turned off. Hence, by conditioning on the first event, we obtain

$$m_1(t) = \int_{x=0}^{\infty} m_1(t|X = x)\gamma e^{-\gamma x} \mathrm{d}x,$$

where

$$\begin{aligned} m_1(t|X = x) &= \frac{\lambda}{\gamma}(1 + m_1(t - x)) + \frac{\alpha}{\gamma} m_0(t - x), \qquad x \leq t, \\ &= 0, \qquad x > t. \end{aligned}$$

So,

$$\begin{aligned} m_1(t) &= \frac{\lambda}{\gamma}\int_{x=0}^{t}(1 + m_1(t - x))\gamma e^{-\gamma x}\mathrm{d}x + \frac{\alpha}{\gamma}\int_{x=0}^{t} m_0(t - x)\gamma e^{-\gamma x}\mathrm{d}x, \\ &= \frac{\lambda}{\gamma}\left(1 - e^{-\gamma t}\right) + \frac{\lambda}{\gamma}\int_{x=0}^{t} m_1(t - x)\gamma e^{-\gamma x}\mathrm{d}x + \frac{\alpha}{\gamma}\int_{x=0}^{t} m_0(t - x)\gamma e^{-\gamma x}\mathrm{d}x, \end{aligned}$$

and the integral equation for $m_0(t)$ is obtained similarly.

b. Transforming the two intergral equations gives

$$\begin{aligned} \mu_1(s) &= \frac{\lambda}{\gamma + s} + \frac{\lambda}{\gamma + s}\cdot\mu_1(s) + \frac{\alpha}{\gamma + s}\cdot\mu_0(s), \\ \mu_0(s) &= \frac{\beta}{\beta + s}\cdot\mu_1(s), \end{aligned}$$

yielding

$$\mu_1(s) = \frac{\lambda(\beta + s)}{s(\alpha + \beta + s)} = \frac{\lambda\beta}{(\alpha + \beta)s} + \frac{\lambda\alpha}{(\alpha + \beta)(\alpha + \beta + s)}.$$

c. Inverting the transform $\mu_1(s)$ gives

$$m_1(t) = \frac{\lambda\beta}{\alpha + \beta}\cdot t + \frac{\lambda\alpha}{(\alpha + \beta)^2}\left(1 - e^{-(\alpha+\beta)t}\right).$$

d. $m_1(t) = \lambda E[O(t)]$.

**Points:**

| 1a | b | c | d | e | 2a | b | c | d | e | 3a | b | c | d | e | 4a | b | c | d |
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| 3 | 1 | 1 | 3 | 2 | 1 | 2 | 3 | 2 | 2 | 1 | 3 | 2 | 1 | 3 | 3 | 3 | 2 | 2 |