TECHNISCHE UNIVERSITEIT EINDHOVEN
Department of Mathematics and Computer Science
Exam Stochastic Processes 2 (2S480) on November 24, 2005, 09.00-12.00.

1. Two workers share an office that contains two telephones. At any time, each worker is either 'working' or 'on the phone'. Each 'working' period of a worker lasts for an exponentially distributed time with rate λ, and each 'on the phone' period lasts for an exponentially distributed time with rate μ. Let $X(t)$ denote the number of workers 'on the phone' at time t.

a. Argue that $\{X(t), t \geq 0\}$ is a continuous time Markov chain and give its transition rates.

b. Suppose that at time 0 there is exactly one worker 'on the phone'. Compute the expected time until both workers are 'working' for the first time.

c. What proportion of time are both workers 'working'?

Suppose now that one of the phones has broken down. Suppose that a worker who is about to use a phone and finds it being used waits until the phone becomes available.

d. What proportion of time are both workers 'working'?

e. The same question as part d, but now suppose that a worker who is about to use a phone but finds it being used begins a new 'working' period.

2. Consider a branching process $\{X_i, i = 0, 1, 2, \ldots\}$, starting with one particle: $X_0 = 1$. The probability generating function of the number of offspring Z of one particle is given by

$$G(r) = E[r^Z] = \frac{1-c}{1-cr}, \qquad |r| \leq 1,$$

where $0 < c < 1$.

a. Compute $E[Z]$, $E[Z(Z-1)]$ and $\mathrm{Var}[Z]$.

b. Show that $P[X_1 = j] = (1-c)c^j, j = 0, 1, 2, \ldots$.

c. Compute $E[X_n]$ for $n = 1, 2, \ldots$.

d. Compute the extinction probability for all possible values of c.

e. Compute $E[r^{X_2}]$, the probability generating function of X_2.

3. Each of n skiers continually, and independently, climbs up and then skis down a particular slope. The time it takes skier i to climb up has distribution F_i with mean μ_i, and it is independent of her time to ski down, which has distribution H_i with mean θ_i, $i = 1, \ldots, n$. Let $N(t)$ denote the total number of times members of this group have skied down the slope by time t. Also, let $U(t)$ denote the number of skiers climbing up the hill at time t.

a. What is $\lim_{t\to\infty} N(t)/t$?

b. Find $\lim_{t\to\infty} E[U(t)]$.

Suppose all F_i are exponential with rate λ and all G_i are exponential with rate μ. Let $X_i(t)$ denote the location of skier i at time t, where state 1 means going up and state 0 going down.

c. Derive the Kolmogorov's forward equations for the transitions probabilities $P_{11}(t)$ and $P_{10}(t)$ of the continuous time Markov chain $\{X_i(t), t \geq 0\}$.

d. Determine $P_{11}(t)$.

e. What is $P[U(t) = k]$? (Hint: $U(t) = \sum_{i=1}^{n} X_i(t)$).

4. Poisson arrivals (with rate λ) join a queue in front of two parallel servers A and B, having exponential service rates μ_A and μ_B, where $\mu_A > \mu_B$. When the system is empty, arrivals go into the fast server A. Otherwise, the head of the queue takes the first free server.

a. What conditions on λ, μ_A and μ_B are necessary to guarantee that the system is stable?

b. Define states and set up the balance equations for the limiting probabilities (do not solve yet).

c. In terms of the probabilities in part b, what is the average number in the system?

d. In terms of the probabilities in part b, what is the probability that an arbitrary arrival will get serviced in the slow server B?

Suppose $\lambda = 3$, $\mu_A = 5$ and $\mu_B = 1$.

e. Solve the balance equations for the limiting probabilities.

Points:

1a	b	c	d	e	2a	b	c	d	e	3a	b	c	d	e	4a	b	c	d	e
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2