TECHNISCHE UNIVERSITEIT EINDHOVEN
Department of Mathematics and Computer Science
Solutions to Exam Stochastic Processes 2 (2S480) on November 24, 2005, 09.00-12.00.

1.

a. The transition rate matrix Q is

$$\begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -\lambda-\mu & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix}.$$

b. Let T_i $(i = 1, 2)$ denote the time, starting from state i, it takes to enter state 0. Then

$$E[T_1] = \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}E[T_2], \qquad E[T_2] = \frac{1}{2\mu} + E[T_1].$$

Solving these equations gives

$$E[T_1] = \frac{2\mu+\lambda}{2\mu^2}.$$

c. The balance equations are:

$$\begin{aligned} 2\lambda P_0 &= \mu P_1, \\ \lambda P_1 &= 2\mu P_2, \end{aligned}$$

from which, together with the normalization equation, follows that

$$P_0 = \left(\frac{\mu}{\lambda+\mu}\right)^2.$$

Alternatively, this can be obtained by noting that both workers are working independently, and the probability that a working is 'working' is $\mu/(\lambda+\mu)$.

d. Now the transition matrix becomes

$$\begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -\lambda-\mu & \lambda \\ 0 & \mu & -\mu \end{pmatrix}.$$

Setting up the balance equations and solving these yields

$$P_0 = \frac{\mu^2}{\mu^2+2\lambda\mu+2\lambda^2}.$$

e. Now only two states are possible, state 0 and 1. Balance of flow gives

$$2\lambda P_0 = \mu P_1,$$

and thus

$$P_0 = \frac{\mu}{\mu + 2\lambda}\,.$$

2.

a. We have

$$E[Z] = G'(1) = \frac{c}{1-c}, \qquad E[Z(Z-1)] = G''(1) = \frac{2c^2}{(1-c)^2},$$

and thus,

$$\mathrm{Var}[Z] = E[Z(Z-1)] + E[Z] - E[Z]^2 = \frac{c}{(1-c)^2}.$$

b.

$$G(r) = \sum_{j=0}^{\infty} P[Z=j]r^j = (1-c)\sum_{j=0}^{\infty} c^j r^j,$$

and hence,

$$P[X_1 = j] = P[Z=j] = (1-c)c^j, \qquad j = 0,1,2,\ldots.$$

c. $E[X_n] = E[Z]^n = \left(\frac{c}{1-c}\right)^n$.

d. The extinction probability $\pi_0 = 1$ for all c for which $E[Z] \leq 1$, and thus for all $0 < c \leq \frac{1}{2}$. If $\frac{1}{2} < c < 1$, then π_0 is the unique root on $(0,1)$ of the equation

$$\pi_0 = G(\pi_0),$$

yielding

$$\pi_0 = \frac{1-c}{c}\,.$$

e.

$$\begin{aligned}
E[r^{X_2}] &= \sum_{j=0}^{\infty} E[r^{X_2}|X_1=j]P[X_1=j] \\
&= \sum_{j=0}^{\infty} E[r^{Z_1+\cdots+Z_j}](1-c)c^j \\
&= \sum_{j=0}^{\infty} G(r)^j(1-c)c^j \\
&= \frac{1-c}{1-cG(r)} = \frac{(1-c)(1-cr)}{1-cr-c(1-c)}\,.
\end{aligned}$$

3.

a. $\lim_{t\to\infty} N(t)/t = \sum_{i=1}^{n} \frac{1}{\mu_i+\theta_i}$.

b. $\lim_{t\to\infty} E[U(t)] = \sum_{i=1}^{n} \frac{\mu_i}{\mu_i+\theta_i}$.

c.

$$\begin{aligned} P'_{11}(t) &= \mu P_{10}(t) - \lambda P_{11}(t), \\ P'_{10}(t) &= \lambda P_{11}(t) - \mu P_{10}(t). \end{aligned}$$

d. Substituting $P_{10}(t) = 1 - P_{11}(t)$ gives

$$P'_{11}(t) = \mu - (\lambda+\mu)P_{11}(t), \quad P_{11}(0) = 1.$$

The solution is

$$P_{11}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\, e^{-(\lambda+\mu)t}, \quad t \geq 0.$$

e.

$$P[U(t) = k] = \binom{n}{k} P_{11}^{k}(t)(1 - P_{11}(t))^{n-k}.$$

4.

a. $\lambda < \mu_A + \mu_B$.

b. The states are n, $n \geq 0$, refering to the number in the system. For $n = 1$, however, we distinguish between state A (server A is busy) and state B (server B is busy). The balance equations are

$$\begin{aligned} \lambda P_0 &= \mu_A P_A + \mu_B P_B, \\ (\lambda+\mu_A)P_A &= \lambda P_0 + \mu_B P_2, \\ (\lambda+\mu_B)P_B &= \mu_A P_2, \\ (\lambda+\mu_A+\mu_B)P_n &= \lambda P_{n-1} + (\mu_A+\mu_B)P_{n+1}, \qquad n \geq 2, \end{aligned}$$

where $P_1 = P_A + P_B$.

c.

$$L = P_A + P_B + \sum_{n=2}^{\infty} nP_n.$$

d. The probability that an arbitrary arrival will get serviced in B is

$$P_A + \frac{\mu_B}{\mu_A + \mu_B} \sum_{n=2}^{\infty} P_n.$$

e.

$$P_0 = \frac{5}{17}, \quad P_A = \frac{9}{68}, \quad P_B = \frac{15}{68}, \quad P_2 = \frac{3}{17}, \quad P_n = \left(\frac{1}{2}\right)^{n-2} \frac{3}{17}.$$

Points:

1a	b	c	d	e	2a	b	c	d	e	3a	b	c	d	e	4a	b	c	d	e
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2