

TECHNISCHE UNIVERSITEIT EINDHOVEN
 Department of Mathematics and Computer Science
 Solutions of exercises Stochastic Processes 2 (2S480) for week 7, 2006.

1. Since, from Example 7.3, $m(t) = e^t - 1$, $0 < t \leq 1$, we obtain upon using the identity $t + E[Y(t)] = \mu(m(t) + 1)$ that $E[Y(1)] = e/2 - 1$.

2. The long-run average cost is

$$\frac{[c + 2c + \dots + (N - 1)c]/\lambda + KNc + \lambda K^2 c/2}{N/\lambda + K} = \frac{c(N - 1)N/2\lambda + KNc + \lambda K^2 c/2}{N/\lambda + K}.$$

3. Say that the system is on at t if $X_{N(t)+1}$, the interarrival time at t , is less than c (and off otherwise). Hence, the proportion of time that $X_{N(t)+1}$ is less than c is

$$\frac{E[\text{on time in a renewal cycle}]}{E[\text{cycle time}]} = \frac{\int_0^c t f(t) dt}{E[X]}.$$

4.

a. This is an alternating renewal process, with the mean off time obtained by conditioning on which machine fails to cause the off period.

$$\begin{aligned} E[\text{off}] &= \sum_{i=1}^3 E[\text{off} | i \text{ fails}] P(i \text{ fails}) \\ &= \frac{1}{5} \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} + 2 \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{3}{2} \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}. \end{aligned}$$

As the on time in a cycle is exponential with rate equal to $\lambda_1 + \lambda_2 + \lambda_3$, we obtain that p , the proportion of time that the system is working is

$$p = \frac{1/(\lambda_1 + \lambda_2 + \lambda_3)}{E[C]},$$

where

$$E[C] = E[\text{cycle time}] = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + E[\text{off}].$$

b. Think of the system as a renewal reward process by supposing that we earn 1 per unit of time that machine 1 is being repaired. Then, r_1 , the proportion of time that machine 1 is being repaired is

$$r_1 = \frac{\frac{1}{5} \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}}{E[C]}.$$

- c. By assuming that we earn 1 per unit time when machine 2 is in a state of suspended animation, shows that, with s_2 being the proportion of time that 2 is in a state of suspended animation,

$$s_2 = \frac{\frac{1}{5} \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{3}{2} \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}}{E[C]}.$$

5. Let T be the time it takes the shuttle to return. Now, given T , X is Poisson with mean λT . Thus,

$$E[X|T] = \lambda T, \quad \text{Var}(X|T) = \lambda T.$$

Consequently,

- a. $E[X] = E[E[X|T]] = \lambda E[T]$.
- b. $\text{Var}(X) = E[\text{Var}(X|T)] + \text{Var}(E[X|T])$ (see Proposition 3.1) $= \lambda E[T] + \lambda^2 \text{Var}(T)$.
- c. Assume that a reward of 1 is earned each time the shuttle returns empty. Then, from the renewal reward theory, r , the rate at which the shuttle returns empty, is

$$\begin{aligned} r &= \frac{P(\text{empty})}{E[T]} \\ &= \frac{\int_0^\infty P(\text{empty}|T=t) f(t) dt}{E[T]} \\ &= \frac{\int_0^\infty e^{-\lambda t} f(t) dt}{E[T]} \\ &= \frac{E[e^{-\lambda T}]}{E[T]}. \end{aligned}$$

- d. Assume that a reward of 1 is earned each time that a customer writes an angry letter. Then, with N_a equal to the number of angry letters written in a cycle, it follows that r_a , the rate at which angry letters are written, is

$$\begin{aligned} r_a &= E[N_a]/E[T] \\ &= \int_0^\infty E[N_a|T=t] f(t) dt / E[T] \\ &= \int_c^\infty \lambda(t-c) f(t) dt / E[T] \\ &= \lambda E[\max\{0, T-c\}] / E[T]. \end{aligned}$$

Since passengers arrive at rate λ , this implies that the proportion of passengers that write angry letters is r_a/λ .

- e. Because passengers arrive at a constant rate, the proportion of them that have to wait more than c will equal the proportion of time that the age of the renewal process (whose event times are the return times of the shuttle) is greater than c . It is thus equal to $1 - F_e(c)$.