TECHNISCHE UNIVERSITEIT EINDHOVEN
Department of Mathematics and Computer Science
Solutions of exercises Stochastic Processes 2 (2S480) for week 8, 2006.

1. Let W_q denote the amount of time that a customer spends in the queue. Then

$$\begin{aligned}
P(W_q \le x) &= \sum_{n=0}^{\infty} P(W_q \le x | n \text{ in system when he arrives}) \left(\frac{\lambda}{\mu}\right)^n \left(1-\frac{\lambda}{\mu}\right) \\
&= \left(1-\frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} P(\text{sum of } n \text{ service times} \le x) \left(\frac{\lambda}{\mu}\right)^n \\
&= \left(1-\frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \sum_{k=n}^{\infty} e^{-\mu x} \frac{(\mu x)^k}{k!} \\
&= \left(1-\frac{\lambda}{\mu}\right) e^{-\mu x} \sum_{k=0}^{\infty} \frac{(\mu x)^k}{k!} \sum_{n=0}^{k} \left(\frac{\lambda}{\mu}\right)^n \\
&= e^{-\mu x} \sum_{k=0}^{\infty} \frac{(\mu x)^k}{k!} \left[1-\left(\frac{\lambda}{\mu}\right)^{k+1}\right] \\
&= e^{-\mu x} \left(e^{\mu x} - \frac{\lambda}{\mu} e^{\lambda x}\right) \\
&= 1 - \frac{\lambda}{\mu} e^{-(\mu-\lambda)x}.
\end{aligned}$$

2. Take the state to be the number of customers at server 1. The balance equations are

$$\begin{aligned}
\mu P_0 &= \mu P_1, \\
2\mu P_j &= \mu P_{j+1} + \mu P_{j-1}, \qquad 1 \le j < n, \\
\mu P_n &= \mu P_{n-1}, \\
1 &= \sum_{j=0}^{n} P_j.
\end{aligned}$$

It is easy to check that the solution to these equations is that all the P_j's are equal, and so $P_j = 1/(n+1)$, $j = 0, \ldots, n$.

3.

a.

$$\begin{aligned}
\lambda P_0 &= \alpha\mu P_1, \\
(\lambda + \alpha\mu) P_n &= \lambda P_{n-1} + \alpha\mu P_{n+1}, \qquad n \ge 1.
\end{aligned}$$

These are exactly the same equations as in the $M/M/1$ with $\alpha\mu$ replcaing μ. Hence,

$$P_n = \left(\frac{\lambda}{\alpha\mu}\right)^n \left(1 - \frac{\lambda}{\alpha\mu}\right), \qquad n \geq 0,$$

and we need the condition $\lambda < \alpha\mu$.

b. If T is the waiting time until the customer first enters service, then by conditioning on the number present when he arrives yields

$$\begin{aligned} E[T] &= \sum_{n=0}^{\infty} E[T|\text{n present}]P_n \\ &= \sum_{n=0}^{\infty} \frac{n}{\mu} P_n \\ &= \frac{L}{\mu}. \end{aligned}$$

Since $L = \sum_{n=0}^{\infty} nP_n$, and the P_n are the same as in the $M/M/1$ with λ and $\alpha\mu$, we have that

$$L = \frac{\lambda}{\alpha\mu - \lambda},$$

and so

$$E[T] = \frac{\lambda}{\mu(\alpha\mu - \lambda)}.$$

c. $P(\text{enters service exactly } n \text{ times}) = (1-\alpha)^{n-1}\alpha$.

d. This is the expected number of services times the mean service time $= 1/(\alpha\mu)$.

e. The distribution is easily seen to be memoryless. Hence, it is exponential with rate $\alpha\mu$.
Note: By Little's law it follows that the expected total time spent in the system is $L/\lambda = 1/(\alpha\mu - \lambda)$.

4. There are four states, namely $0, 1_A, 1_B, 2$. The balance equations are

$$\begin{aligned} 2P_0 &= 2P_{1_B}, \\ 4P_{1_A} &= 2P_0 + 2P_2, \\ 4P_{1_B} &= 4P_{1_A} + 4P_2, \\ 6P_2 &= 2P_{1_B}, \\ 1 &= P_0 + P_{1_A} + P_{1_B} + P_2. \end{aligned}$$

This yields

$$P_0 = \frac{3}{9}, \quad P_{1_A} = \frac{2}{9}, \quad P_{1_B} = \frac{3}{9}, \quad P_2 = \frac{1}{9}.$$

a. $P_0 + P_{1_B} = \frac{2}{3}$.

b. By conditioning upon whether the state was 0 or 1_B when he entered we get that the desired probability is given by

$$\frac{1}{2} + \frac{1}{2} \cdot \frac{2}{6} = \frac{2}{3}.$$

c. $P_{1_A} + P_{1_B} + 2P_2 = \frac{7}{9}$.

d. Again, condition on the state when he enters to obtain

$$\frac{1}{2}\left[\frac{1}{4} + \frac{1}{2}\right] + \frac{1}{2}\left[\frac{1}{4} + \frac{2}{6} \cdot \frac{1}{2}\right] \frac{7}{12}.$$

This could also have been obtained from a. and b. by Little's formula $W = \frac{L}{\lambda_a}$. That is,

$$W = \frac{\frac{7}{9}}{2 \cdot \frac{2}{3}} = \frac{7}{12}.$$