TECHNISCHE UNIVERSITEIT EINDHOVEN
Department of Mathematics and Computer Science
Solutions of exercises Stochastic Processes 2 (2S480) for week 9, 2006.

1. The state space can be taken to consist of states $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, where the ith component of the state refers to the number of customers at server i, $i = 1,2$. The balance equations are

$$\begin{aligned} 2P_{0,0} &= 6P_{0,1}, \\ 8P_{0,1} &= 4P_{1,0} + 4P_{1,1}, \\ 6P_{1,0} &= 2P_{0,0} + 6P_{1,1}, \\ 10P_{1,1} &= 2P_{0,1} + 2P_{1,0}, \end{aligned}$$

and the normalization equation is

$$P_{0,0} + P_{0,1} + P_{1,0} + P_{1,1} = 1.$$

Solving these equations gives $P_{0,0} = 1/2$, $P_{0,1} = 1/6$, $P_{1,0} = 1/4$ and $P_{1,1} = 1/12$.

a. $P_{1,1} = 1/12$.

b. $W = \frac{L}{\lambda_a} = \frac{P_{0,1}+P_{1,0}+2P_{1,1}}{2(1-P_{1,1})} = 7/22$.

c. $\frac{P_{0,0}+P_{0,1}}{1-P_{1,1}} = 8/11$.

2. The state is the pair (i,j), $i = 0,1$, $0 \leq j \leq N$, where i signifies the number of customers in service and j the number in orbit. The balance equations are

$$\begin{aligned} (\lambda + j\theta)P_{0,j} &= \mu P_{1,j}, \qquad j = 0,\dots,N, \\ (\lambda + \mu)P_{1,j} &= \lambda P_{0,j} + (j+1)\theta P_{0,j+1}, \qquad j = 0,\dots,N-1, \\ (\lambda + \mu)P_{1,N} &= \lambda P_{0,N}. \end{aligned}$$

c. $1 - P_{1,N}$.

d. The average number of customers in the system is

$$L = \sum_{i,j} (i+j)P_{i,j}.$$

Hence, the average time that an entering customer spends in the system is $W = L/\lambda(1 - P_{1,N})$, and the average time that an entering customer spends in orbit is $W - 1/\mu$.

3.

a. State n, $n \geq 0$, refers to the number of customers in the system, where for $n = 1$ we distinguish between state A (server A is busy) and state B (server B is busy). The balance equations are

$$\begin{aligned} \lambda P_0 &= \mu_A P_A + \mu_B P_B, \\ (\lambda + \mu_A) P_A &= \alpha \lambda P_0 + \mu_B P_2, \\ (\lambda + \mu_B) P_B &= (1-\alpha) \lambda P_0 + \mu_A P_2, \\ (\lambda + \mu_A + \mu_B) P_n &= \lambda P_{n-1} + (\mu_A + \mu_B) P_{n+1}, \qquad n \geq 2, \end{aligned}$$

where $P_1 = P_A + P_B$.

b.

$$L = P_A + P_B + \sum_{n=2}^{\infty} n P_n.$$

The average number of idle servers is $2P_0 + P_A + P_B$.

c. Let β denote the probability that an arbitrary arrival will get serviced in A. Then

$$\beta = \alpha P_0 + P_B + \frac{\mu_A}{\mu_A + \mu_B} \sum_{n=2}^{\infty} P_n.$$

Alternatively, β can be computed by using Little's law. We have

$$W_Q = \frac{L_Q}{\lambda},$$

where

$$L_Q = \sum_{n=2}^{\infty} (n-2) P_n.$$

Then β follows from the relation

$$W = \frac{L}{\lambda} = W_Q + \beta \frac{1}{\mu_A} + (1-\beta) \frac{1}{\mu_B}.$$

4. States are $0, 1, 1', 2, 2', \ldots, k-1, (k-1)', k, k+1, \ldots$ with the following interpretation

$$\begin{aligned} 0 &= \text{system is empty}, \\ n &= n \text{ in system and server is working}, \\ n' &= n \text{ in system and server is idle}, \qquad n = 1, 2, \ldots, k-1. \end{aligned}$$

The balance equations are

$$\begin{aligned}
\lambda P_0 &= \mu P_1, \\
(\lambda+\mu) P_0 &= \mu P_2, \\
\lambda P_{n'} &= \lambda P_{(n-1)'}, \qquad n = 1, \ldots, k-1, \\
(\lambda+\mu) P_k &= \lambda P_{(k-1)'} + \lambda P_{k-1} + \mu P_{k+1}, \\
(\lambda+\mu) P_n &= \lambda P_{n-1} + \mu P_{n+1}, n > k.
\end{aligned}$$

b.

$$W_Q = \frac{k-1}{\lambda} P_0 + \sum_{n=1}^{k-1} \left[\frac{k-1-n}{\lambda} + \frac{n}{\mu} \right] P_{n'} + \sum_{n=1}^{\infty} \frac{n}{\mu} P_n,$$

or via Little's law,

$$W_Q = \frac{L_Q}{\lambda},$$

where

$$L_Q = \sum_{n=1}^{k-1} n P_{n'} + \sum_{n=1}^{\infty} (n-1) P_n.$$

c. $\lambda < \mu$.